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Hybrid Quantum Annealing for Solving Nonlinear Differential Equations

with Spectral Collocation

ABSTRACT FOCUS

We introduce a hybrid quantum-classical approach where **Chebyshev spectral collocation** discretizes the problem into algebraic equations, which are then transformed into **QUBO models**. This method combines classical control of nonlinearity with quantum annealer speedup for core residual minimization.

Motivation & Problem Statement

Bridging the gap between classical numerical analysis and NISQ hardware constraints

Why Hybrid Quantum?

- **Nonlinear Challenges**
  Classical solvers face dense Jacobian matrices and poor scaling when handling strong nonlinearities in PDEs.
- **NISQ Constraints**
  Current quantum hardware (NISQ) is limited by shallow circuit depths, noise, and low qubit counts, making full quantum solvers impractical.
- **Strategic Offloading**
  The goal is to offload **only the residual minimization** to the quantum processor, leveraging its optimization capabilities.

KEY INSIGHT

"Keep classical strengths (control, accuracy) and use quantum only where it might offer a specific advantage."

Problem Formulation

We consider a second-order nonlinear boundary value problem (BVP) on the domain $[0, 2]$:

$$\frac{d}{dx}\left[a(x,u)u'\right] = f(x)$$

Subject to boundary conditions:

$u(0) = \alpha$ $u(2) = \beta$

Test Cases Investigated:

1. `a(x,u) = 1` - Constant Coefficient (Linear)
2. `a(x,u) = x + 1` - Variable Coefficient
3. `a(x,u) = u + 1` - Nonlinear (Primary Challenge)

Classical Background

Chebyshev Spectral Collocation Method

Discretization Nodes

We define $N+1$ Chebyshev-Gauss-Lobatto (CGL) nodes x_j on $[-1,1]$, then map to physical domain $[0,2]$.

$$x_j = \cos\left(\frac{\pi j}{N}\right), \quad j = 0, \dots, N$$

** Clustering near boundaries enhances resolution where gradients are steep.*

CORE CONCEPT

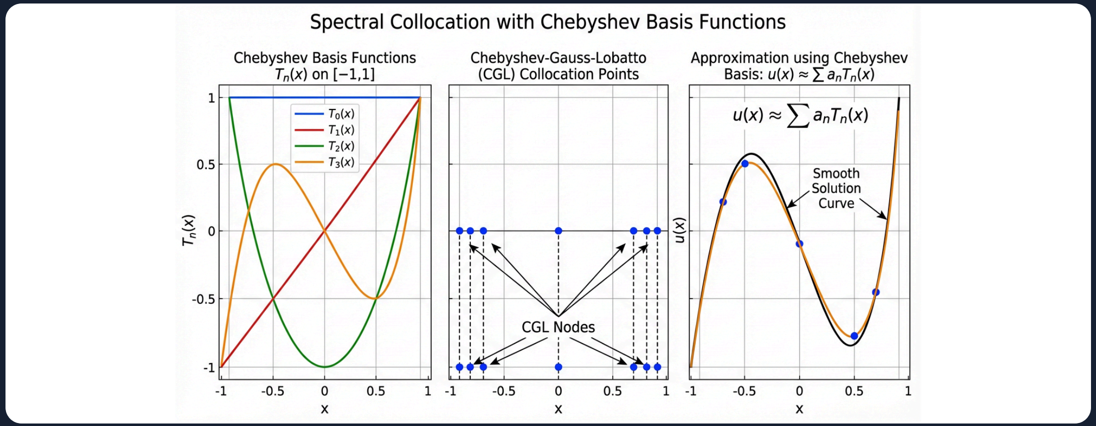
Residual Construction

Using the differentiation matrix D, the continuous BVP becomes an algebraic system. For a solution vector $\mathbf{u}$:

$$R(\mathbf{u}) = D\,\mathrm{diag}(a(\mathbf{u})) \circ (D\mathbf{u}) - \mathbf{f}$$

- $\circ$: Element-wise mult.
- $\mathbf{f}$: Forcing term

Basis & Convergence



Why Spectral Methods?

- **Global Basis:** Solution $u(x) \approx \sum a_n T_n(x)$ uses global polynomials.
- **Exp. Convergence:** Error $O(c^{-N})$ for smooth solutions.

Optimization Goal

Minimize the squared norm of the residual vector:

$$L(\mathbf{u}) = \|R(\mathbf{u})\|_2^2 \to 0$$

* Minimization step offloaded to quantum backend.

Hybrid Strategy

Combining classical control with quantum residual minimization

Classical Process   Quantum Process

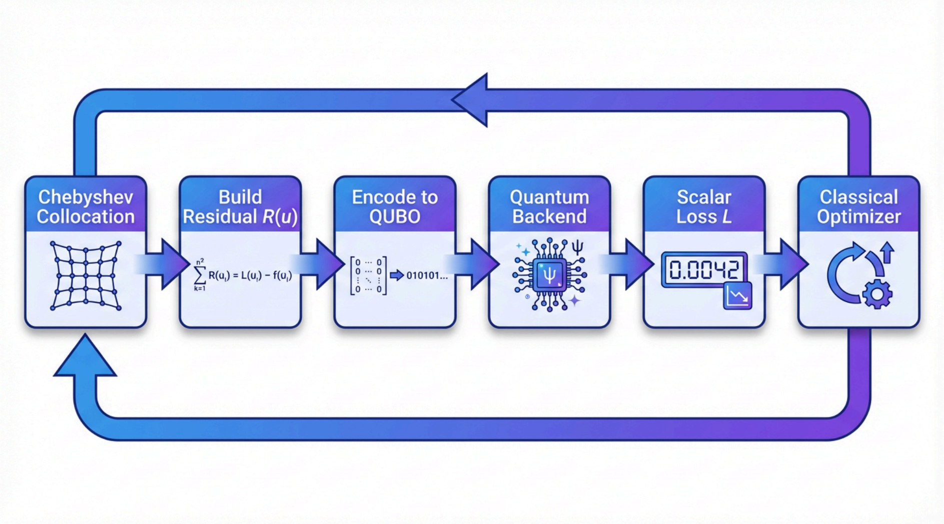
Classical Role

- Constructs **CGL nodes** & diff. matrices
- Manages boundary & nonlinearity (Picard)
- Executes outer loop (BFGS)

Quantum Role

- Minimizes residual norm $L(\theta)$
- Estimates inner products via measurement
- **No Jacobian** calculation

Full Pipeline View



ENCODING STRATEGY

Gate-Based (VQA)
$L(\theta) = \langle\psi|H|\psi\rangle$

Annealer (Focus)
$\min E_{QUBO}(\mathbf{x}, \mathbf{z})$

CLASSICAL 01
Discretize
Map PDE to algebraic equations via Chebyshev.

↓

CLASSICAL 02
Formulate R(u)
Construct residual vector & loss function.

↓

QUANTUM 03
Minimize L
Map to QUBO & find low-energy state.

↓

CLASSICAL 04
Update
Refine parameters & iterate if needed.

QUBO Formulation

Mapping Differential Equations to Quantum Annealers

From Residual to CUBO

The squared residual loss $L(\mathbf{u}) = \|R(\mathbf{u})\|_2^2$ contains nonlinear terms up to $O(x^3)$ when discretized, yielding a Cubic Unconstrained Binary Optimization (CUBO) problem.

$$E_{CUBO}(\mathbf{x}) = \sum_{i,j,k} Q_{ijk} x_i x_j x_k + \sum_{i,j} Q_{ij} x_i x_j + \sum_i h_i x_i$$

ANNEALER REQUIREMENT

Reduction to QUBO

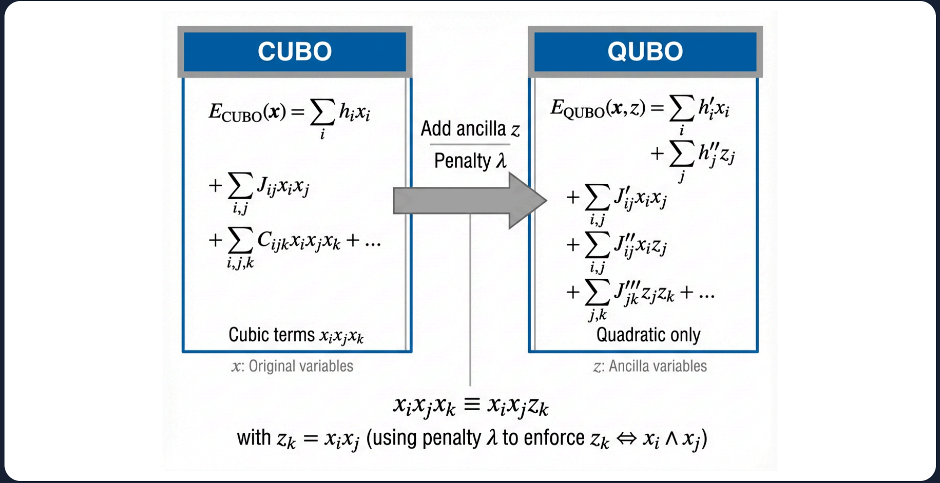
Annealers minimize quadratic forms. We reduce cubic terms by introducing auxiliary variables z_k and penalty λ:

$$x_i x_j x_k \equiv x_i x_j z \quad \text{subject to} \quad z = x_k$$

$$E_{QUBO}(\mathbf{x}, \mathbf{z}) = E'_{CUBO}(\mathbf{x}, \mathbf{z}) + \lambda P(\mathbf{x}, \mathbf{z})$$

** λ enforces the constraint ensuring mathematical equivalence.*

Transformation Process



Implementation Details

- </> **Binary Encoding:** Chebyshev coefficients are encoded into binary variables with appropriate scaling.
- **Iterative Loop:** Hybrid Picard iteration where each step solves a linearised QUBO problem.
- **Backend:** Automatski annealer used to minimize the resulting QUBO energy.

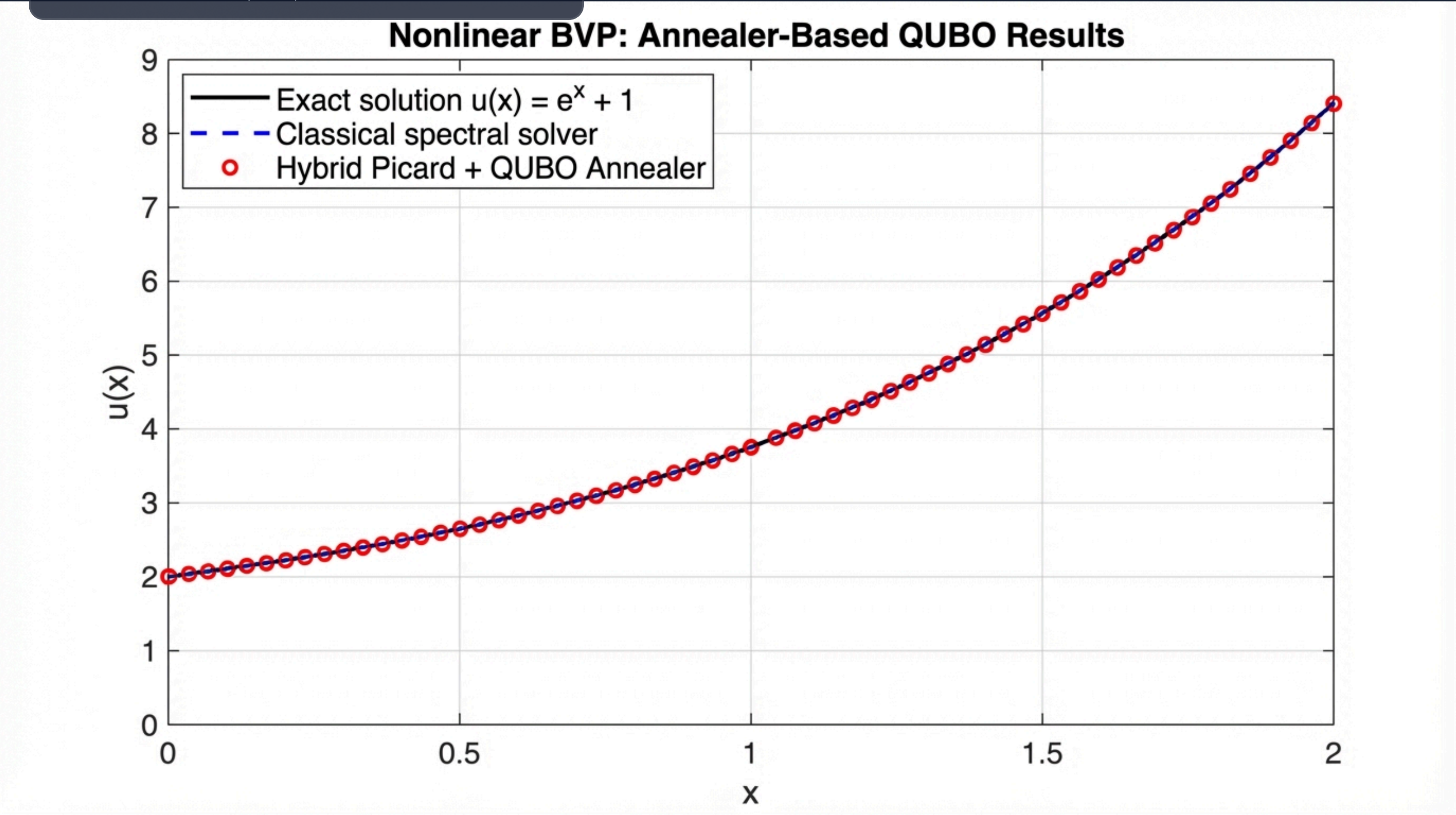
Implementation & Results

Annealer-based QUBO Performance on Nonlinear BVP

N = 50 Nodes | Automatski Annealer | Picard + QUBO

Solution Comparison: $u(x)$ vs x

Nonlinear Case: $a(x,u) = u + 1$



ANNEALER PERFORMANCE

High Fidelity

The hybrid QUBO solver closely tracks the exponential growth of the exact solution, significantly outperforming initial VQA attempts on gate-based emulators.

CONVERGENCE BEHAVIOR

Stable Iterations

Picard iteration coupled with QUBO minimization converged stably to the correct nonlinear profile.

SETUP DETAILS

- Discretized via Chebyshev Collocation
- Binary encoding of coefficients
- Penalty tuning crucial for constraints

Comparison of Approaches

Gate-Based VQA vs. Annealer-Based QUBO Performance Analysis

GATE-BASED (VQA)	CRITERIA	ANNEALER (QUBO)
Continuous Parameters Variational parameters θ optimized via gradient descent. `Cost:` $\langle\psi(\theta)\|H\|\psi(\theta)\rangle$	</> ENCODING VS	**Discrete Binary** Fixed binary variables representing coefficients. `Cost:` $\min E_{QUBO}(\mathbf{x})$
Non-Convex Landscape Prone to **Barren Plateaus** (vanishing gradients) as depth increases.	OPTIMIZATION VS	✓ Robust **Quadratic Energy** Native tunneling helps escape local minima in quadratic forms.
Limited Precision Exhibited interior bias; constrained by noise and ansatz expressivity.	ACCURACY VS	🏆 Winner **High Fidelity** Matched exact solution closely; successfully captured nonlinear exponential growth.

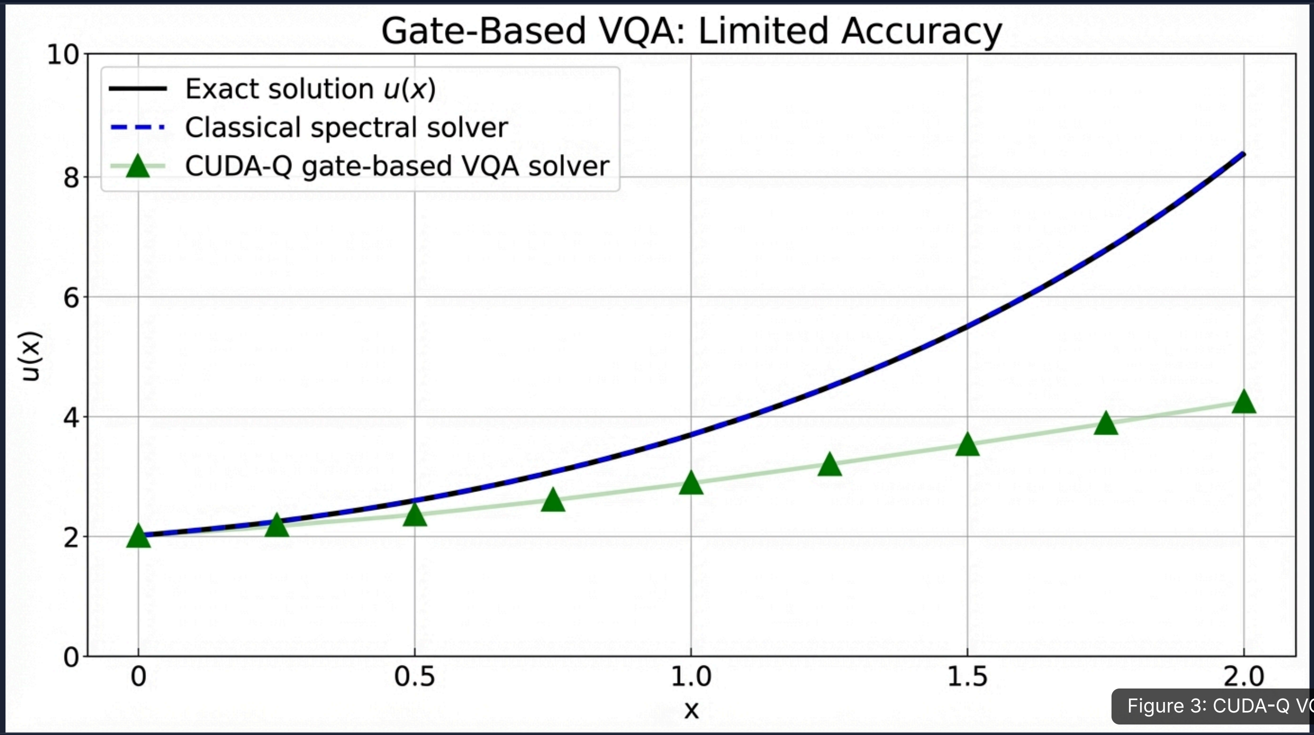


Figure 3: CUDA-Q VQA Deviation

Visual Evidence

The plot (Figure 3) reveals why gate-based VQA struggles:

- Significant interior deviation from exact solution (black line)
- Inability to fully capture rapid exponential growth
- Limited by ansatz expressivity on NISQ hardware

→ *Contrast with Annealer (Slide 6) which followed the curve precisely.*

Key Findings

Summary of experimental outcomes and methodological validation

01

Spectral Accuracy

Classical solver reproduced the exact solution with $\sim 10^{-12}$ error, validating Chebyshev collocation as a robust discretization baseline.

02

Annealer Superiority

Annealer-based QUBO produced the best quantum solution, capturing nonlinear exponential growth where gate-based methods struggled with the landscape.

03

VQA Limits

Gate-based solvers showed interior bias and difficulty converging on the non-convex residual cost function, diverging from the exact solution.

04

Shallow Circuit Advantage

Counter-intuitively, shallower circuits (depth-1) outperformed deeper ones (depth-3), indicating that noise resilience is currently more critical than ansatz expressivity on NISQ hardware.

05

Viable "Residual-Only" Model

The "Classical Control + Quantum Residual" architecture successfully minimized quantum overhead by offloading only the core optimization task, proving practical for near-term devices.

Conclusions & Future Work

Pathways to Practical Quantum Advantage

Key Takeaways

Annealer Success

Annealer-based QUBO formulations currently outperform gate-based VQA for nonlinear BVPs, offering stable convergence and higher fidelity in capturing exponential growth.

HYBRID VIABILITY

The "Classical Control + Quantum Residual" architecture minimizes quantum overhead, proving effectively that residual offloading is a viable path for NISQ devices.

HARDWARE ROADMAP

Near-term: Annealers for quadratic optimization.
Long-term: Gate-based devices for generalized spectral methods as coherence times improve.

Next Steps

QUBO Scaling Analysis

Investigate how ancilla qubit requirements scale with discretization size N during the CUBO-to-QUBO reduction.

Meshfree Methods (RBF)

Extend the solver to Radial Basis Functions (RBFs) to handle irregular domains where spectral collocation is limited.

Warm-Start Protocols

Develop a unified scheme using annealer solutions to initialize (warm-start) gate-based VQA parameters.

PRIMARY PUBLICATION —— 2025

"A Hybrid Quantum–Classical Spectral Solver for Nonlinear Differential Equations"

MDPI Algorithms 18(11), 678 • doi.org/10.3390/a18110678

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