Spectral methods for nonlinear dispersive PDEs

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Outline

- Introduction
- Multi-domain method, compactified domains
- nonlinear Schrödinger equations
- Korteweg-de Vries equations
- Helmholtz equations
- Benjamin-Ono equations
- Outlook

Nonlinear dispersive PDEs

- nonlinearity against dispersion, stable structures (*solitons*)
- rapid oscillations (*dispersive shocks*)
- blow-up (loss of regularity in finite time), limit of applicability of the model
- most complete results for integrable equations, results generic?

Hopf equation

- hyperbolic conservation law for $u(x,t)$, initial data $u_0(x)$

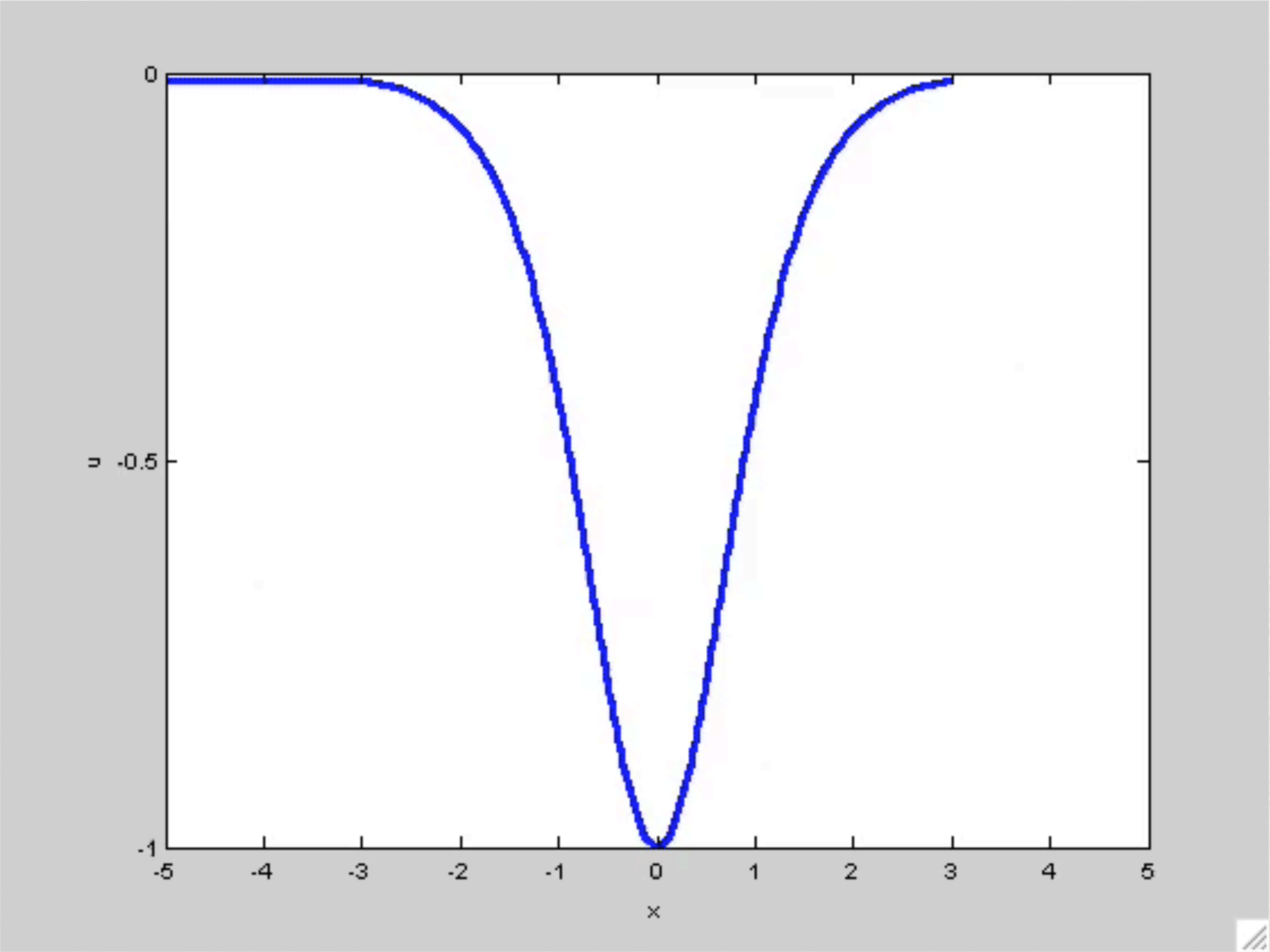
  $u_t + 6uu_x = 0, \; u(x,0) = u_0(x)$

- solution with the method of characteristics

  $u(x,t) = u_0(\xi), \; x = 6tu_0(\xi) + \xi$

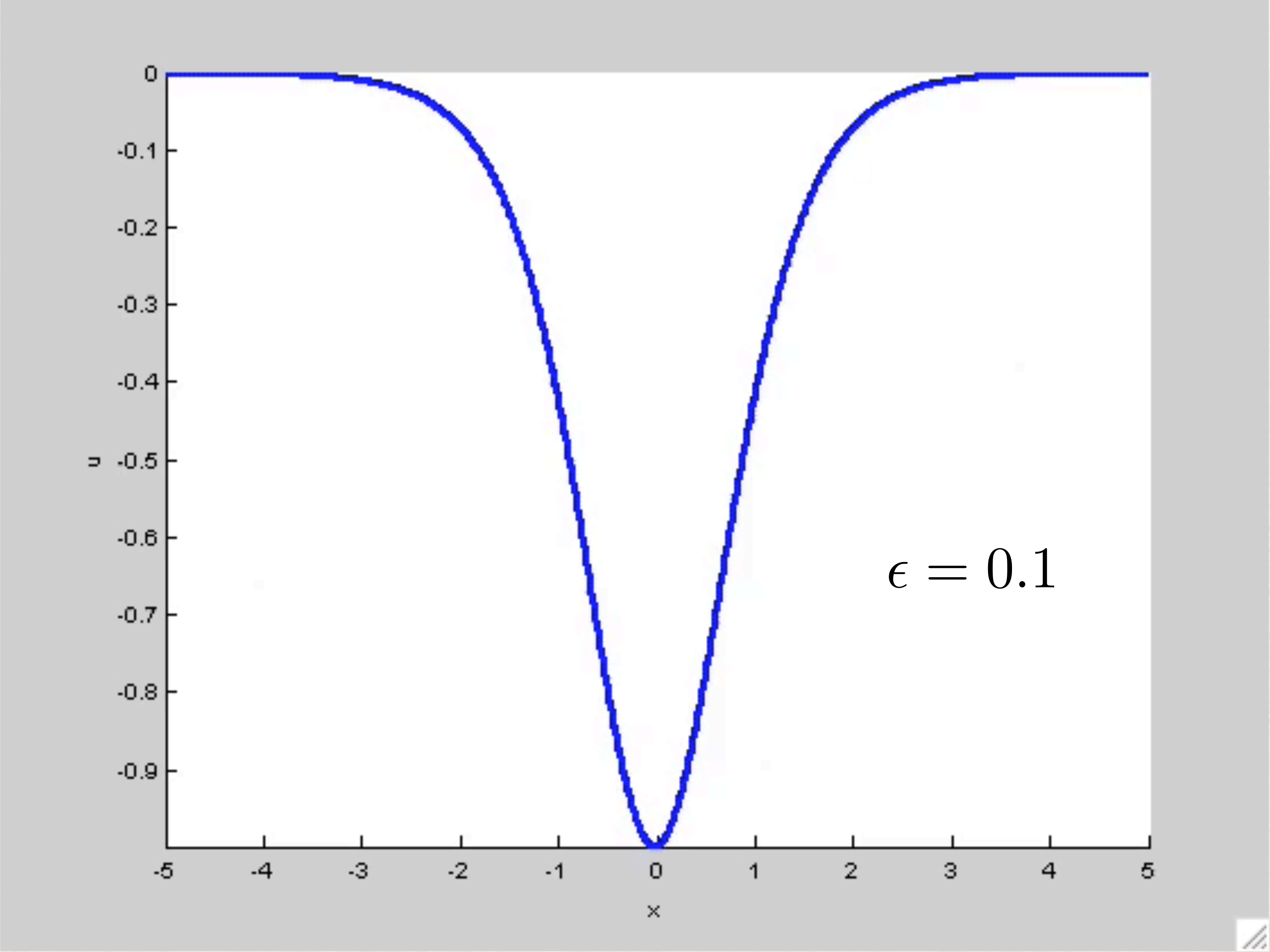
- critical time $t_c = \dfrac{1}{\min_{\xi \in \mathbb{R}} [-6u_0'(\xi)]}$, gradient catastrophe,
  $t > t_c$: solution multivalued (shock)

Example: $u_0 = -\mathrm{sech}^2 x$



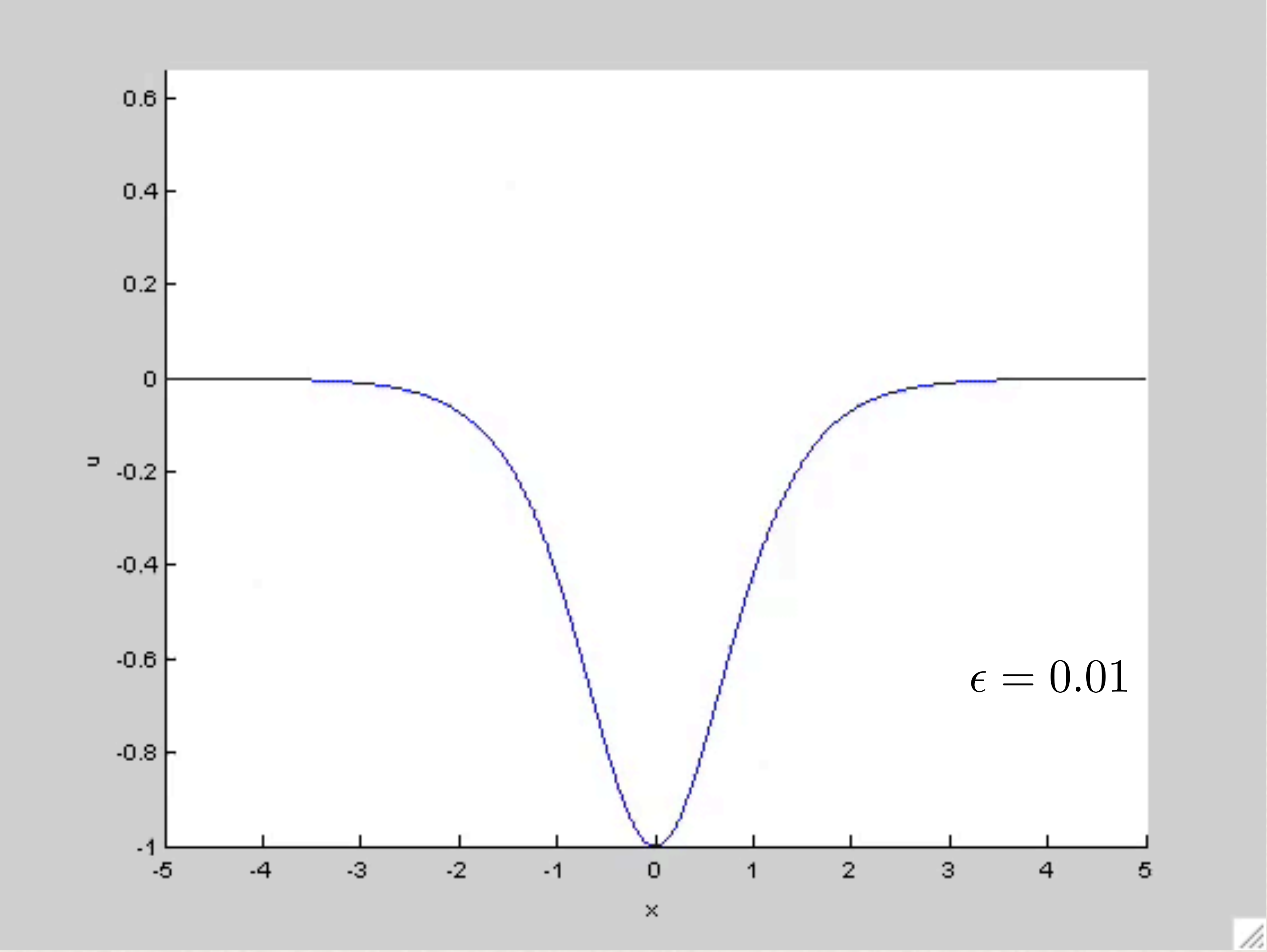
Dissipative regularization

$$u_t + 6uu_x = \epsilon u_{xx}$$

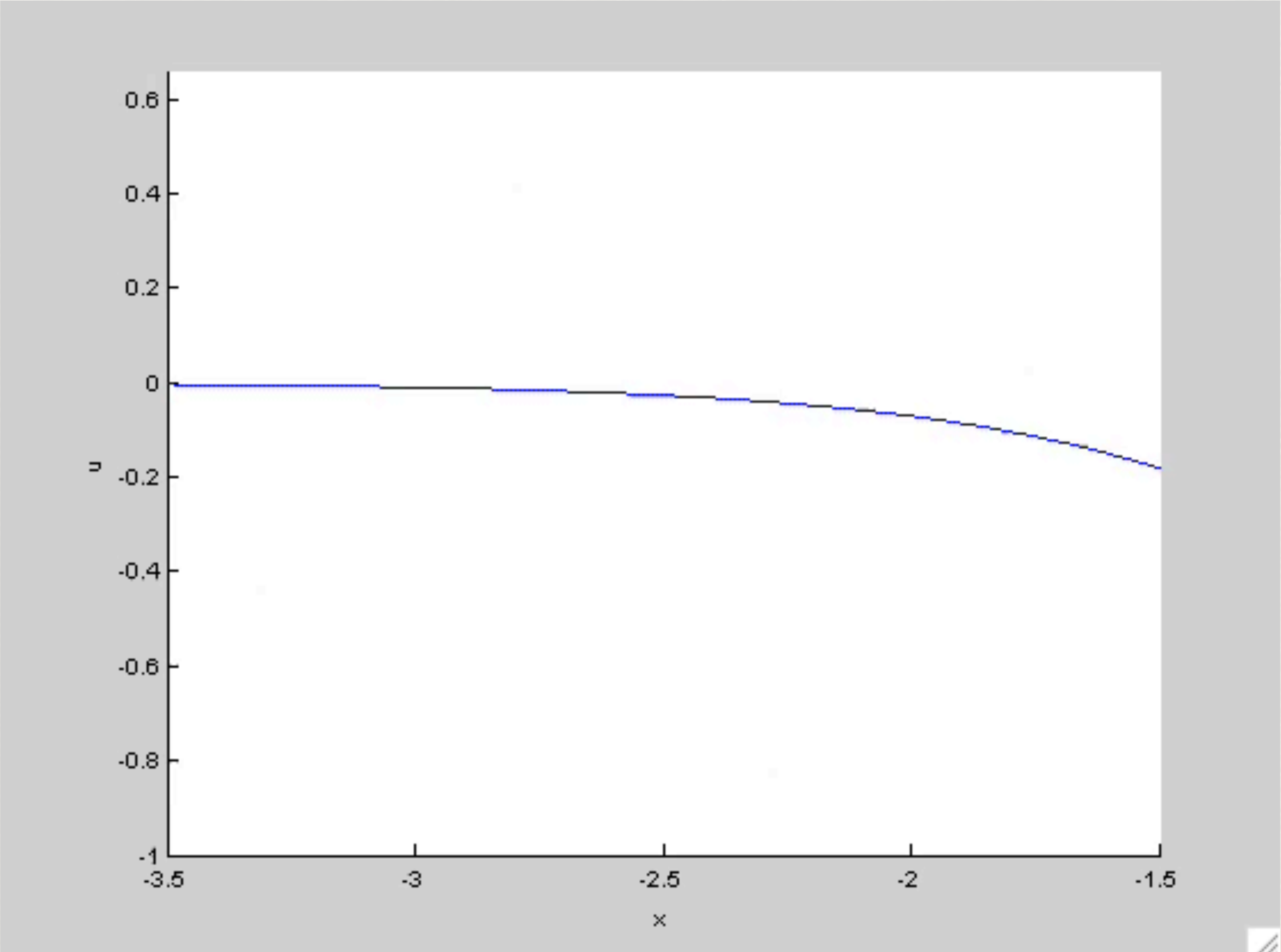


Korteweg-de Vries equation

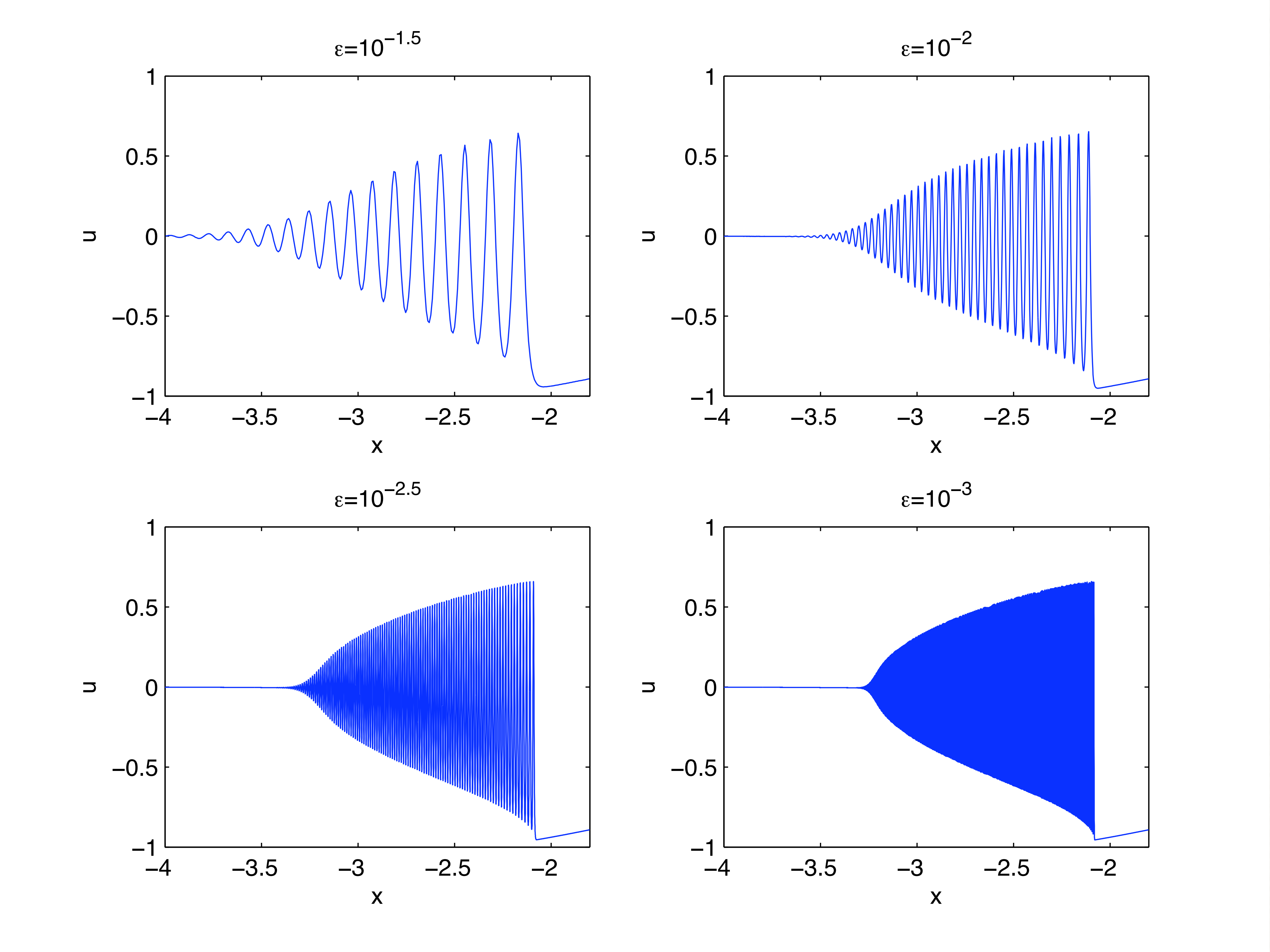
$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0 \qquad u_0 = -\mathrm{sech}^2 x$$



Zoom in: dispersive shock



Different values of ε



$t = 0.4$

Blow-up

- generalized KdV equation

$$u_t + u^p u_x + \epsilon^2 u_{xxx} = 0, \quad p \in \mathbb{N}$$

- $$u_t + \epsilon^2 u_{xxx} = 0$$

  (linear) and

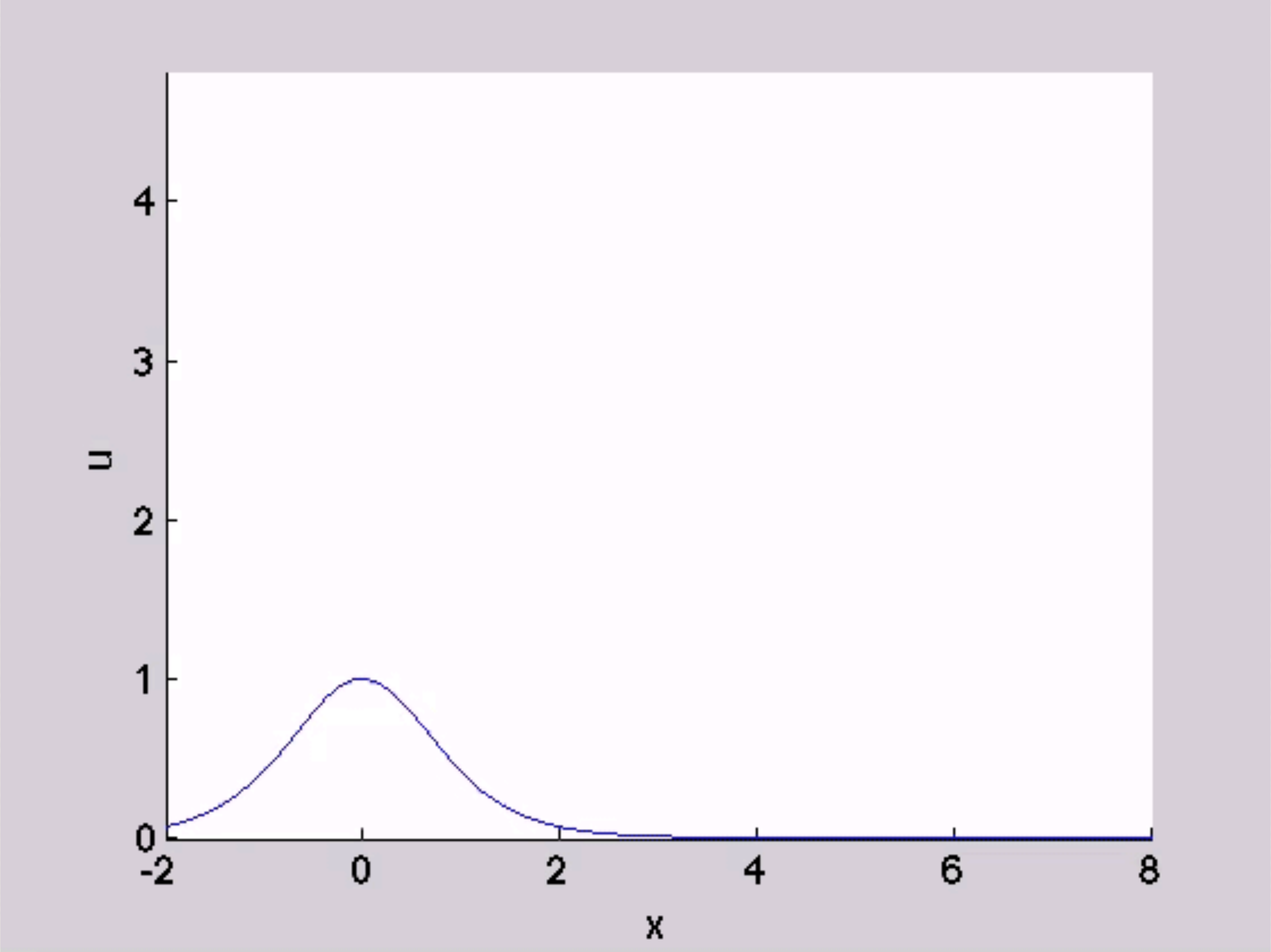
  $$u_t + u^p u_x = 0$$

  (shocks) do not have blow-up of the L_∞ norm of u

- for $p < 4$: global existence in time,
  for $p = 4$: finite time blow-up (Martel, Merle, Raphaël: rescaled soliton),
  for $p > 4$: finite time blow-up, no theory yet.

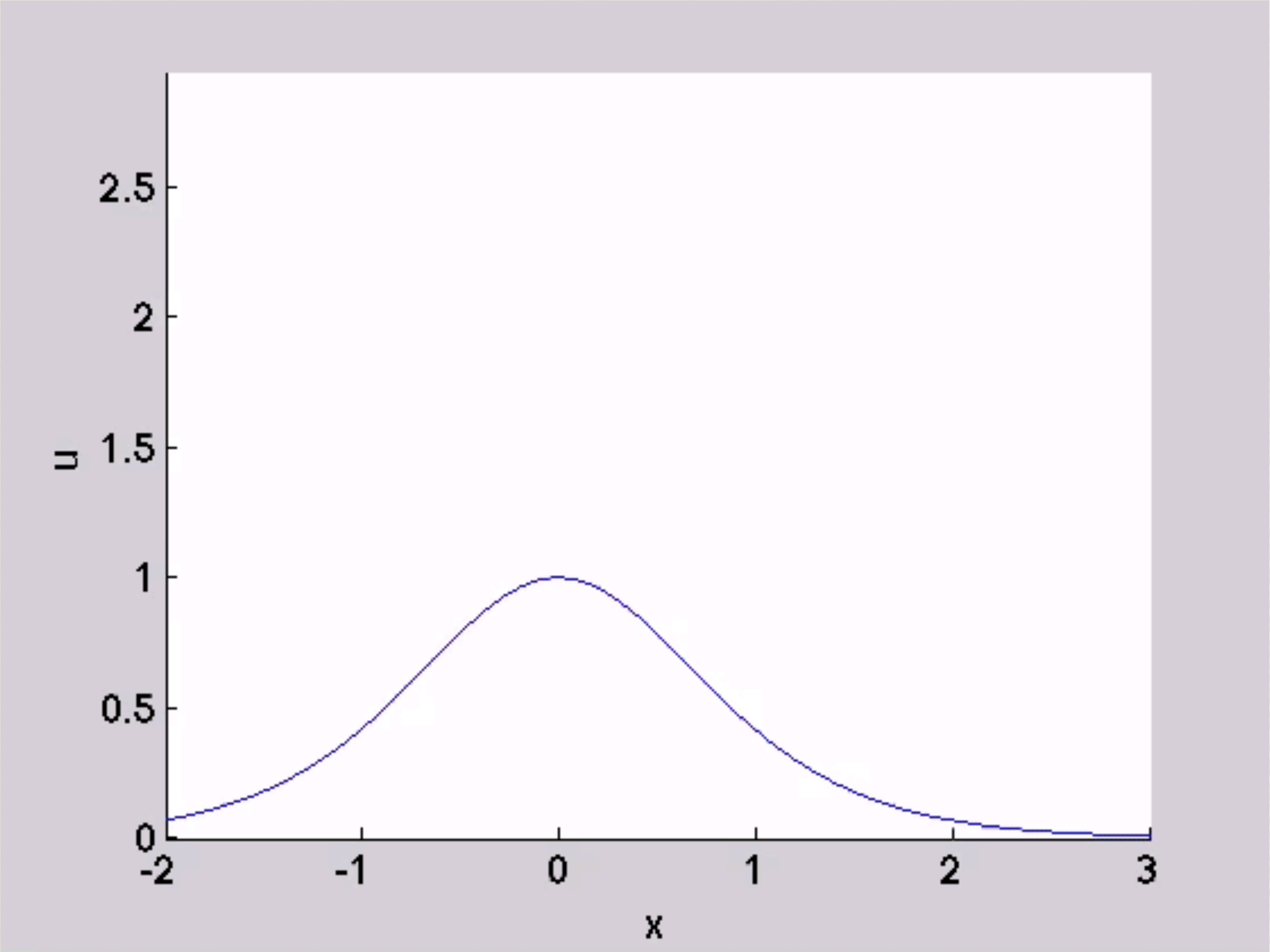
gKdV, small dispersion

$$u_0 = \operatorname{sech}^2 x, \quad \epsilon = 0.1 \qquad n = 4$$



gKdV, small dispersion

$$u_0 = \text{sech}^2 x, \quad \epsilon = 0.01 \qquad n = 4$$



Applied Mathematical Sciences

Christian Klein
Jean-Claude Saut

Nonlinear Dispersive Equations

Inverse Scattering and PDE Methods

Springer

Spectral methods

- first success of spectral methods: Orszag 1971, Orr-Sommerfeld instability
- periodic problems: discrete Fourier series via *fast Fourier transform (fft)*
- exponential decrease of the Fourier coefficients for analytic functions: exponential decrease of the numerical error due to truncation of the series: *spectral convergence*
- non-periodic problems: polynomial interpolation

Polynomial interpolation

- Chebychev collocation points (Runge phenomenon)

$$l_j = \cos\left(\frac{j\pi}{N}\right), \quad j = 0, \dots, N, \quad N \in \mathbb{N}$$

- Lagrange polynomial $p(l)$ of order N for $f : [-1, 1] \mapsto \mathbb{R}$, $p(l_j) = f(l_j)$, $j = 0, \dots, N$

- approximation of the derivative: $f'(l_j) \approx p'(l_j) = \sum_{i=0}^{N} D_{ji} f(l_i)$, $j = 0, \dots, N$; D: *differentiation matrix*

- spectral convergence for smooth $f(l)$

- Clenshaw-Curtis algorithm:

$$\int_{-1}^{1} f(l)dl \approx \int_{-1}^{1} p(l)dl = \sum_{n=0}^{N} w_n f(l_n)$$

Collocation method

- Chebychev polynomials

$$T_n(l) = \cos(n \arccos(l)), \quad n = 0, 1, \dots$$

- collocation method: $f(l) \approx \sum_{n=0}^{N} a_n T_n(l)$, on collocation points:

$$f(l_j) = \sum_{n=0}^{N} a_n T_n(l_j), \quad j = 0, \dots, N$$

relation to *fft* (*fast cosine transformation*, not precompiled in Matlab)

- spectral coefficients a_n, decrease exponentially with n for smooth $f(l)$
- recurrence formula for Chebyshev polynomials, division by l in coefficient space

$$T_{n+1}(l) + T_{n-1}(l) = 2lT_n(x), \quad n = 1, 2, \dots$$

Multidomain method

- spectral method of 'infinite order', but cond$(D^2) = 0(N^4)$; multi-domains to keep N small

- interval $[x_l, x_r]$ mapped to $[-1, 1]$:

$$x = x_l \frac{1+l}{2} + x_r \frac{1-l}{2}, \quad l \in [-1, 1]$$

- compactified exterior domains (CED): $s = 1/x$ local coordinate,

$$u_{xx} = s^4 u_{ss} + 2s^3 u_s$$

  singular for $s = 0$ (compactification with spectral methods first used by Grosch, Orszag 1977, popular in astrophysics)

Schrödinger equations

- one dimensional Schrödinger equation

$$i\partial_t u + \partial_{xx} u + V(|u|^2, x)u = 0$$

- linear case: quantum mechanics, quantum semiconductors, in electromagnetic wave propagation, in seismic migration, lowest order one-way approximation (paraxial wave equation) to the Helmholtz equation, Fresnel equation in optics, underwater acoustics

- $V = \pm 2|u|^2$: focusing (+) or defocusing (−) cubic nonlinear Schrödinger equation (NLS), completely integrable (Zakharov-Shabat)

- NLS: modulation of waves in hydrodynamics, nonlinear optics and plasma physics, Bose-Einstein condensates

M. Birem and C. Klein, Multidomain spectral method for Schrödinger equations, Adv. Comp. Math., 42(2), 395-423 DOI 10.1007/s10444-015-9429-9 (2016)

Choice of numerical approach

- (piecewise) smooth functions: use spectral methods
- rapidly decreasing or periodic functions: Fourier (additional advantage: diagonal differentiations matrices, well conditioned)
- solutions with algebraic decrease, compact support or a finite number of discontinuities: polynomial interpolation
- Galilei invariance: $u(t,x)$ NLS solution, so is

$$\hat{u}(x,t) = u(x-ct,t)e^{icx/2-ic^2t/4},$$

$c \in \mathbb{R}$ finite speed

Rogue waves

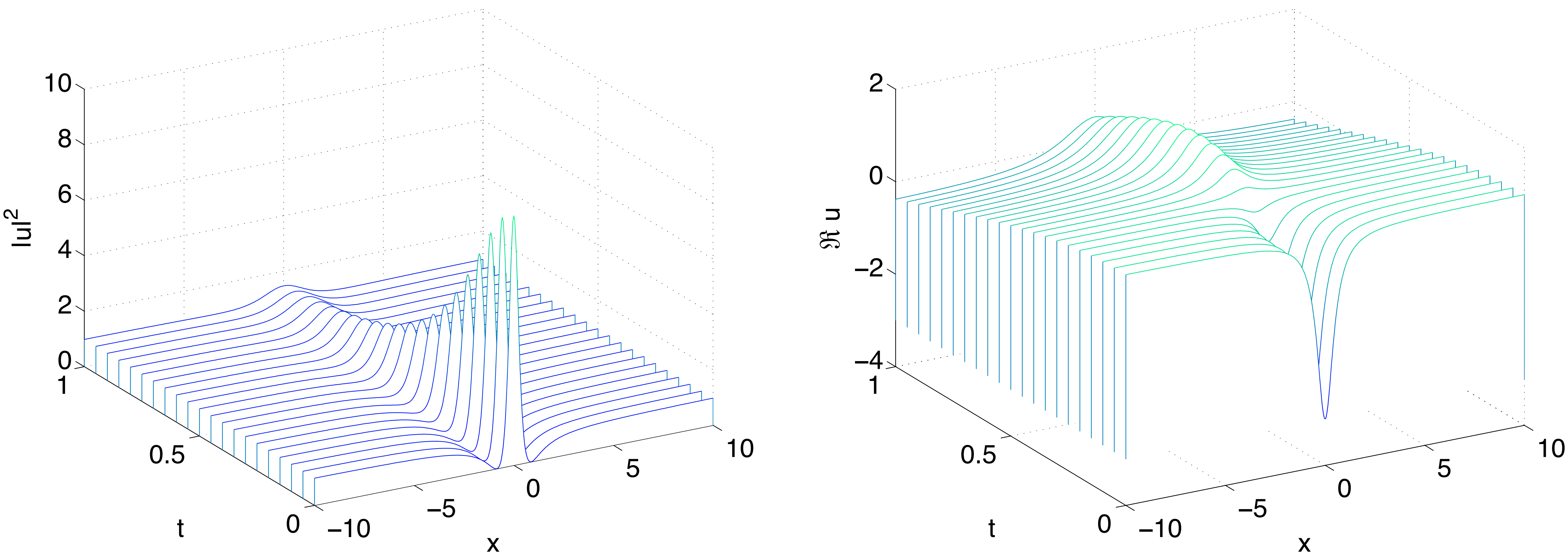
- Giant waves on deep water, more than twice the average wave height
- NLS solutions?

Peregrine solution

- exact solution

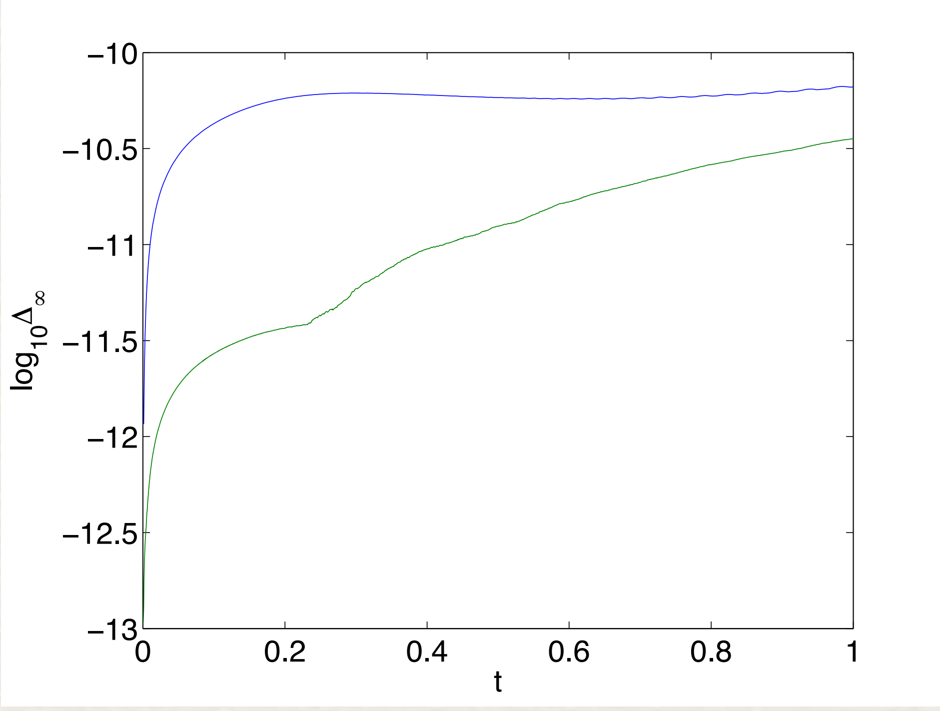
$$u_{Per} = \left(1 - \frac{4(1+4it)}{1+4x^2+16t^2}\right) e^{2it}$$

- $|u|$ asymptotically decaying to 1 (both in x and t), maximum three times the asymptotic value

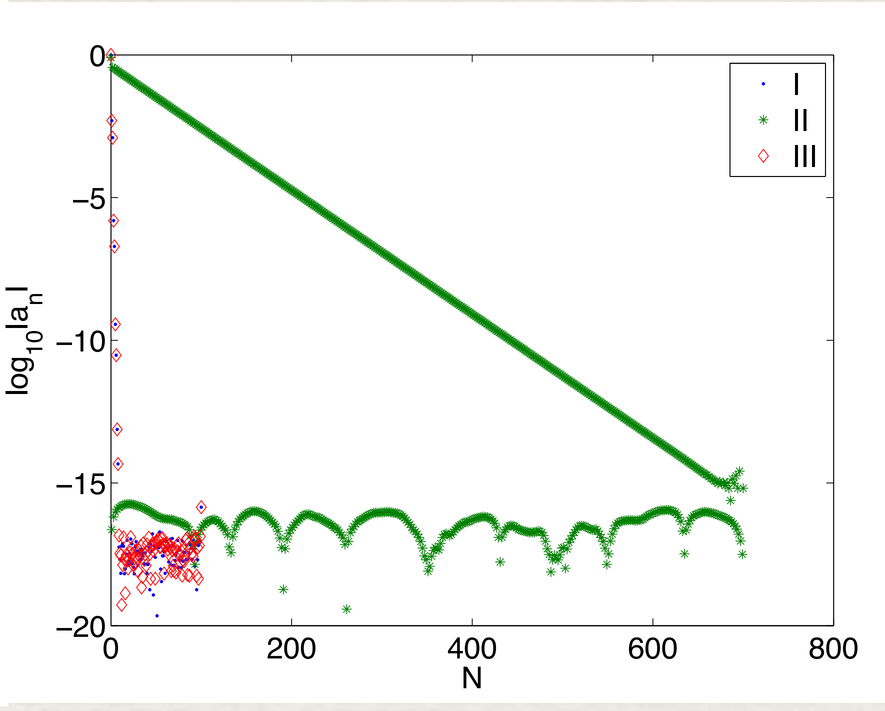


Fourth oder method

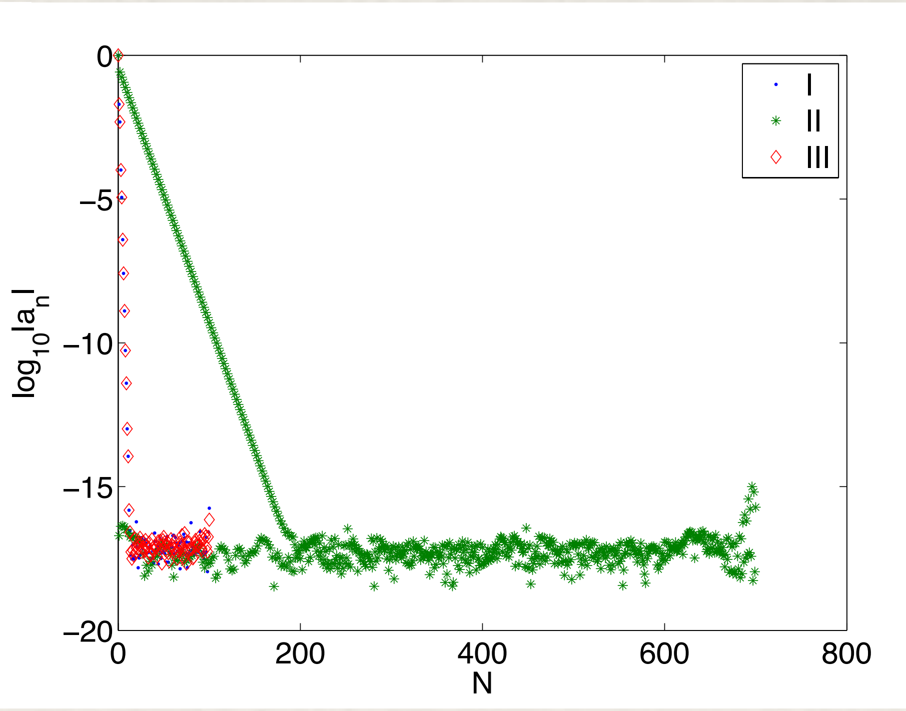
- $x_r = -x_l = 10$
- $N^I = N^{III} = 50,\ N^{II} = 700$
- $N_t = 1000, 2000$



$t = 0$



$t = 1$



„Benjamin-Feir instability"

- linearization: $u = u_{Per}(1+\tilde{v})$

$$i\tilde{v}_t + \tilde{v}_{xx} + 2(\ln u_{Per})_x \tilde{v}_x + 4|u_{Per}|^2 \Re\tilde{v} = 0,$$

numerically problematic for $u_{Per} \approx 0$

- $u_{Per} \to e^{2it}$ for $x \to \infty$ or $t \to \infty$, $\tilde{v} = \alpha + i\beta$

$$\alpha_t + \beta_{xx} = 0, \quad -\beta_t + \alpha_{xx} + 4\alpha = 0$$

- Fourier transform in x, eigenvalues

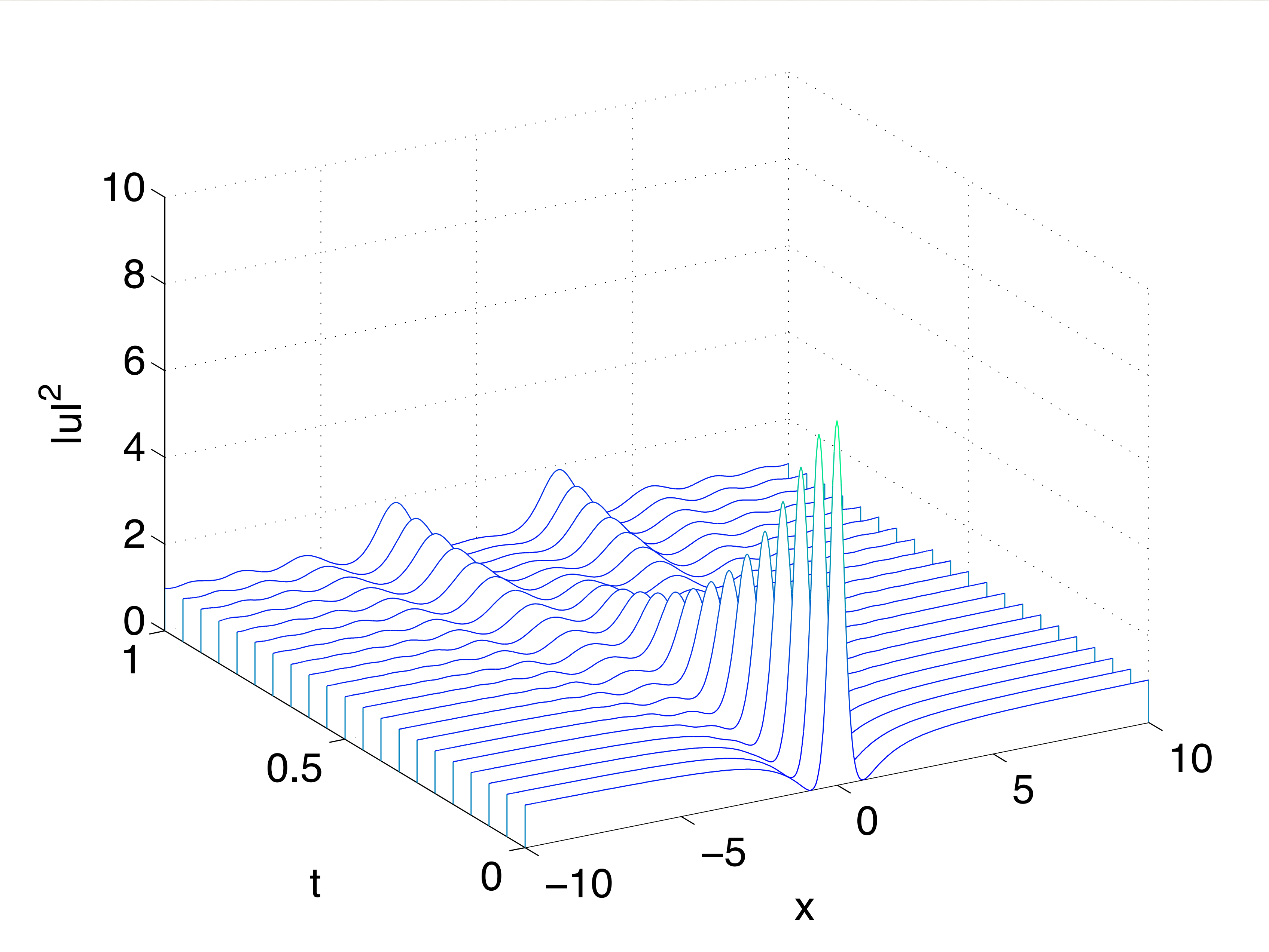
$$\lambda_{1,2}(k) = \pm|k|\sqrt{4-k^2}$$

modulational instability

NLS, localized perturbation

- Gaussian perturbation

$$u(x,0) = u_{Per}(x,0) + 0.1\exp(-x^2)$$



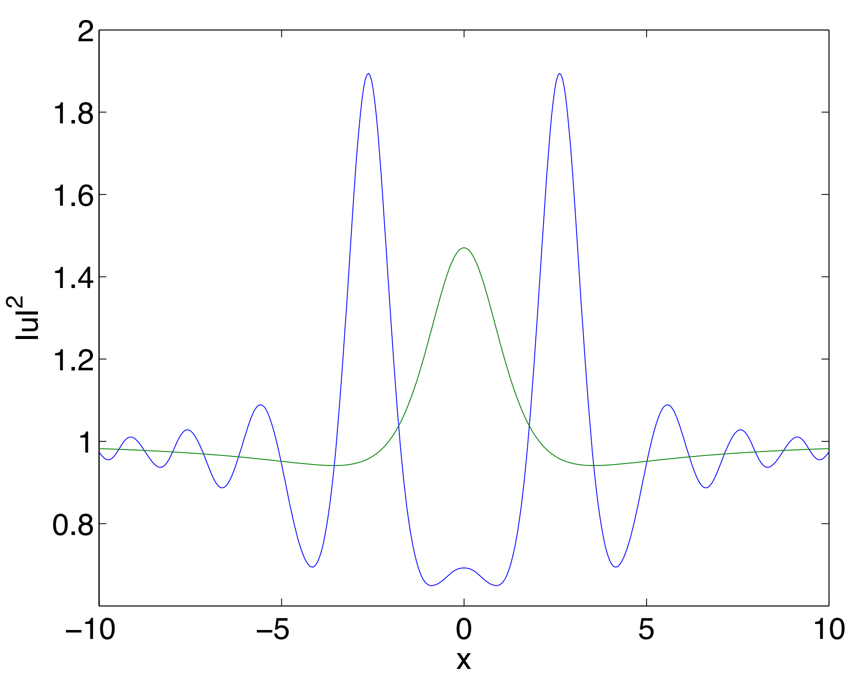
Localized perturbation

- conserved quantity

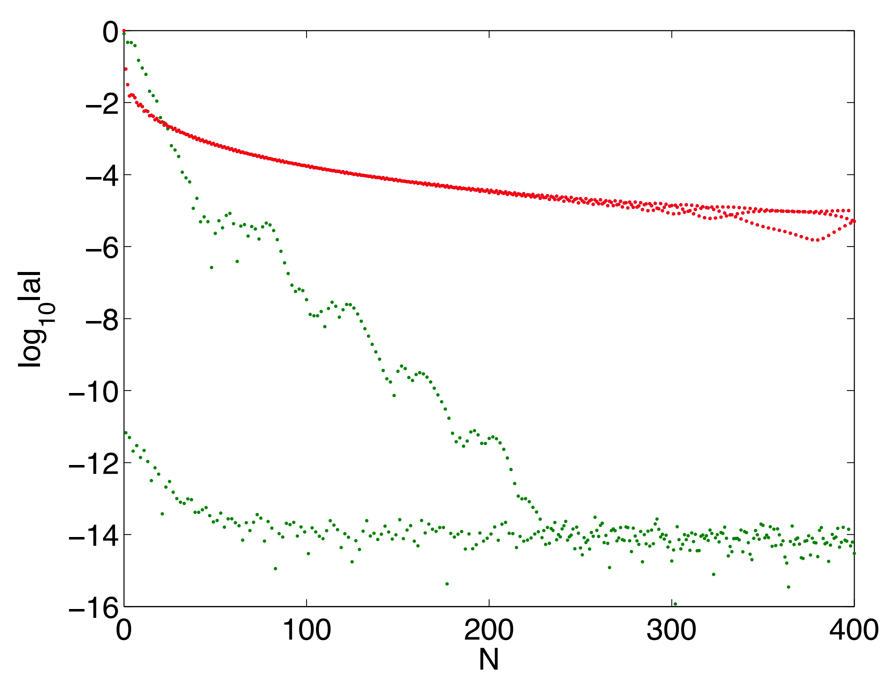
$$E = \frac{1}{2}\int_{\mathbb{R}} \left\{ |u_x|^2 - |u|^2(|u^2|-1) \right\} dx$$

relative conservation better than 10^{-3}

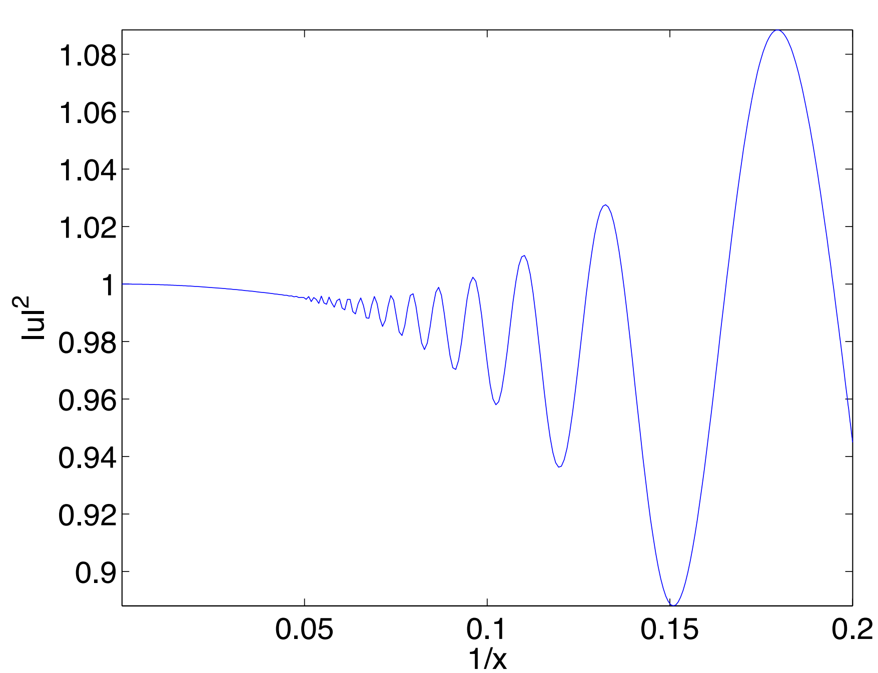
t=1



Chebychev coefficients



compactified zone



Transverse stability

C. Klein, N. Stoilov, Numerical study of the transverse stability of the Peregrine solution, Stud Appl Math. 145 (2020) 36–51. https://doi.org/10.1111/sapm.12306

- 2d NLS

$$i\partial_t u + \partial_{xx} u + \partial_{yy} u + 2|u|^2 u = 0$$

- hypothesis: periodic in y (this includes data rapidly decreasing in y)
- Fast Fourier transform (FFT) techniques in y (diagonal differentiation matrices), $x \in \mathbb{R}$ as before (2 domains, one compactified)
- inexact 4th order splitting technique (linear step integrated with IRK4) fully explicit,

$$\mathcal{L}_+ \mathcal{L}_- (K_1 + K_2) = 2i\Delta U(t_n),$$

where $\mathcal{L}_\pm = \hat{1} - h(0.25i \pm 0.25/\sqrt{3})\Delta$

- perturbation localised in x and y, $t \leq 0.5$

$$u(x, y, 0) = u_{Per} + 0.1\exp(-(x+1)^2 - y^2)$$

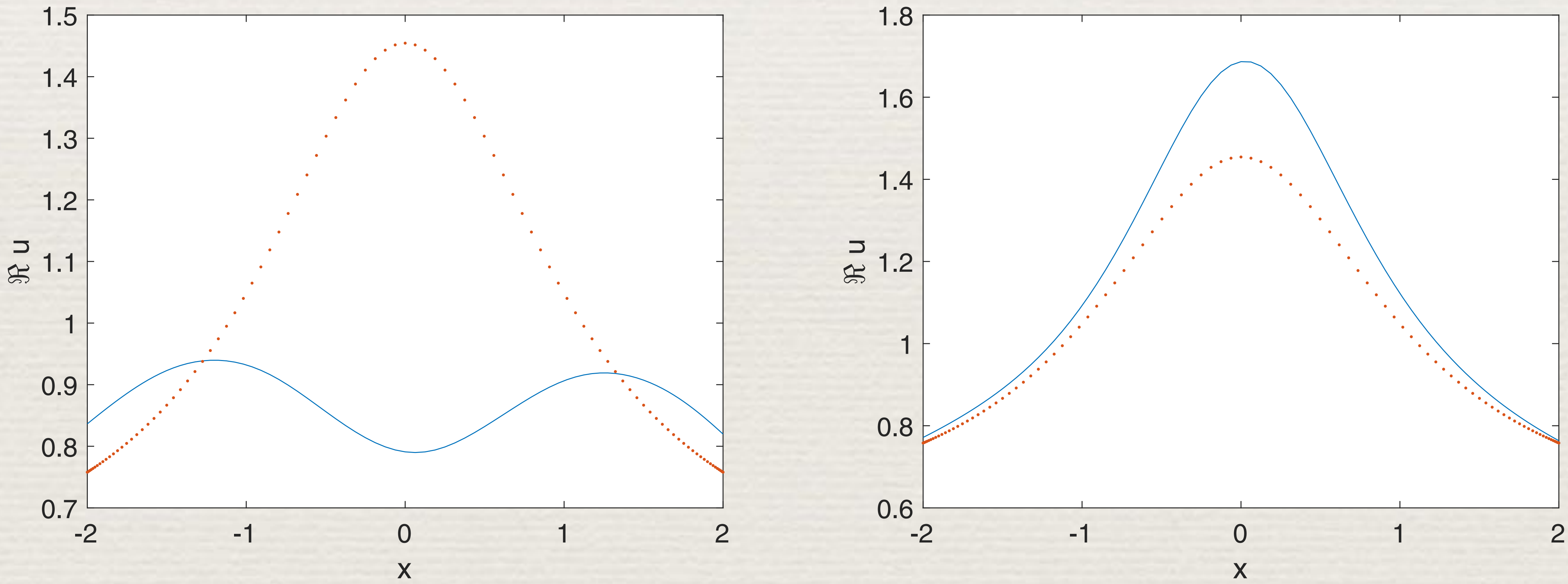


FIGURE 4 Real part of the solution to the 2D NLS equation for the initial data $u(x, y, 0) = u_{Per}(x, t_0) + 0.1\exp(-(x+1)^2 - y^2)$ for $t = 0.5$, on the left for $y = 0$, on the right for $y = 1.6199$; the Peregrine solution for the same time is shown as a dotted line

- perturbation localised in x and y, $t \leq 0.5$

$$u(x,y,0) = u_{Per} + 0.1\exp(-x^2 - y^2)$$

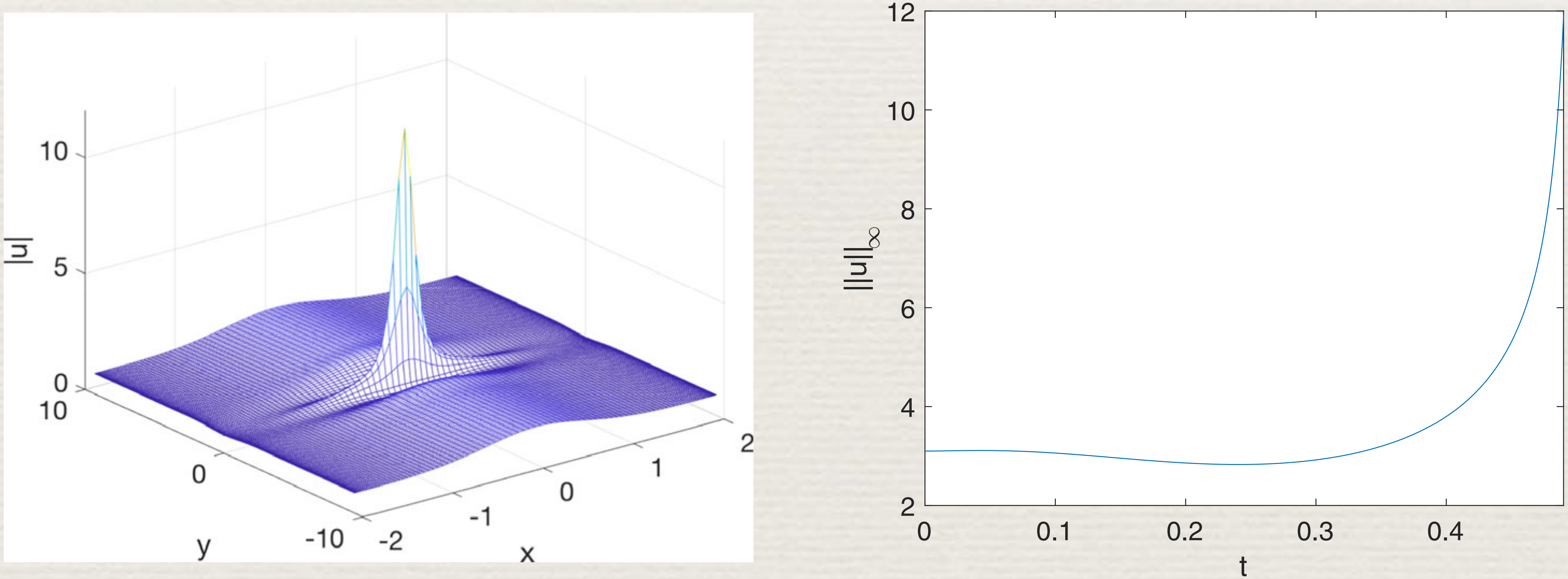


FIGURE 8 Solution to the 2D NLS equation for the initial data $u(x, y, 0) = u_{Per}(x, t_0) - 0.1\exp(-x^2 - y^2)$ for $t = 0.49$ on the left and the L^∞ norm of the solution in dependence of time on the right

Generalized KdV equations

C. Klein, N. Stoilov, Spectral approach to Korteweg-de Vries equations on the compactified real line, App. Num. Math., https://doi.org/10.1016/j.apnum.2022.02.015

- $$u_t(x,t) + u_{xxx}(x,t) + u(x,t)^{p-1} u_x(x,t) = 0, \quad p = 2, 3, \ldots$$

- inverse scattering if Faddeev decay condition holds,

$$\int_{\mathbb{R}} (1 + |x|) |u_0(x)| dx < \infty,$$

- exotic blow-up

- compactification

$$x = c \tan \frac{\pi l}{2}, \quad l \in [-1, 1], \quad c = const$$

- boundary conditions

$$u(l,t)\Big|_{l=1} = 0, \quad u(l,t)\Big|_{l=-1} = 0, \quad u_l(l,t)\Big|_{l=-1} = 0.$$

$$u(x,0) = \frac{1}{(1+x^2)^a}, \quad a = \frac{1}{2}, 1.$$

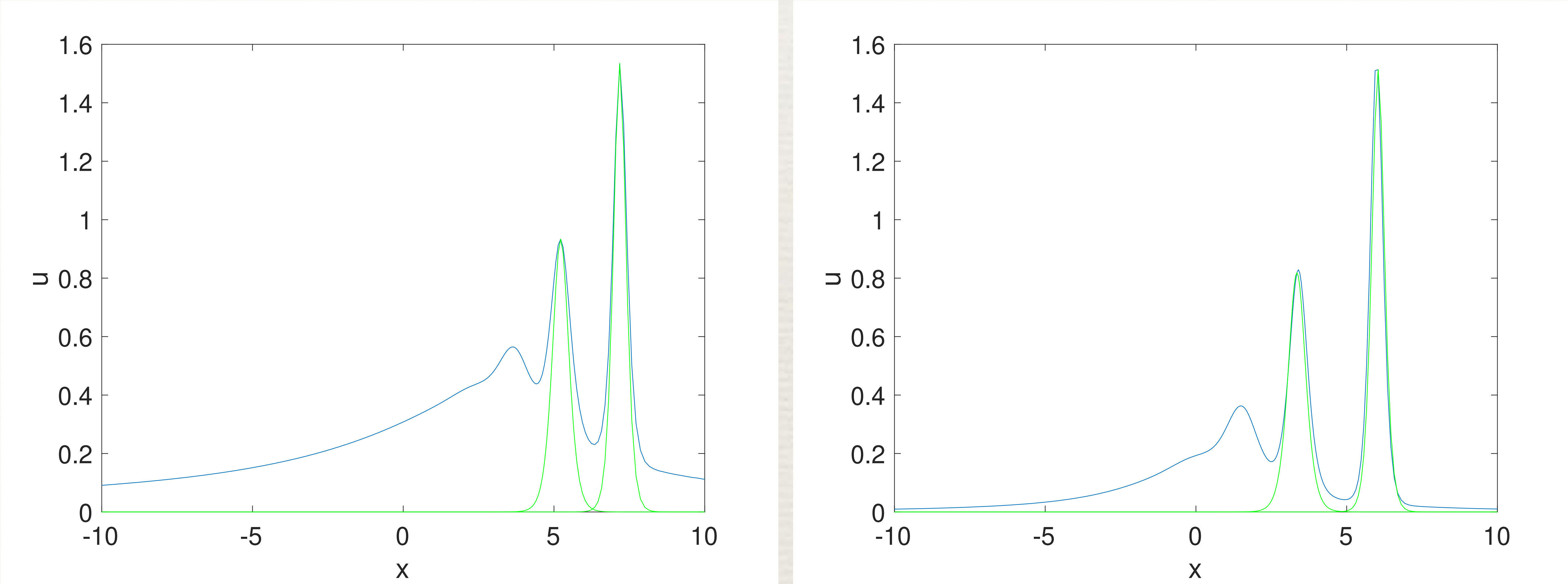


FIGURE 10. Solution to the KdV equation (1) with $p = 2$ for the initial data (22) for $t = 10$, on the left for $a = 1/2$, on the right for $a = 1$; in green fitted solitons (4).

Maxwell equations

- wave equation for each component of the electric field, vector Helmholtz equation after Fourier transform in time

$$\nabla \times \nabla \times \mathbf{E} - \omega^2 \epsilon(r,\omega)\mathbf{E} = 0$$

$\epsilon(r,\omega)$ piecewise constant function

- axial symmetry, spherical coordinates, $\mathbf{E} = (E^\rho(\rho,\theta), E^\theta(\rho,\theta), E^\phi(\rho,\theta))$

$$\rho E^\theta_{\rho\theta} - E^\rho_{\theta\theta} + \cot(\theta)E^\theta - \epsilon(\omega,\rho)\rho^2\omega^2 E^\rho - \cot(\theta)E^\rho_\theta + E^\theta_\theta + \rho\cot(\theta)E^\theta_\rho = 0,$$

$$\rho E^\theta_{\rho\rho} - E^\rho_{\rho\theta} + \epsilon(\omega,\rho)\rho\omega^2 E^\theta + 2E^\theta_\rho = 0,$$

$$\rho^2 E^\phi_{\rho\rho} + E^\phi_{\theta\theta} + E^\phi\left(-\csc(\theta)^2 + \rho^2\omega^2\epsilon(\omega,\rho)\right) + \cot(\theta)E^\phi_\theta + 2\rho E^\phi_\rho = 0,$$

E^ϕ decouples, can be put equal to zero in the axisymmetric case

Sommerfeld condition

C. Klein, N. Stoilov, Multidomain spectral approach with Sommerfeld condition for the Maxwell equations, J. Comp. Phys., https://doi.org/10.1016/j.jcp.2021.110149

- no incoming radiation from infinity

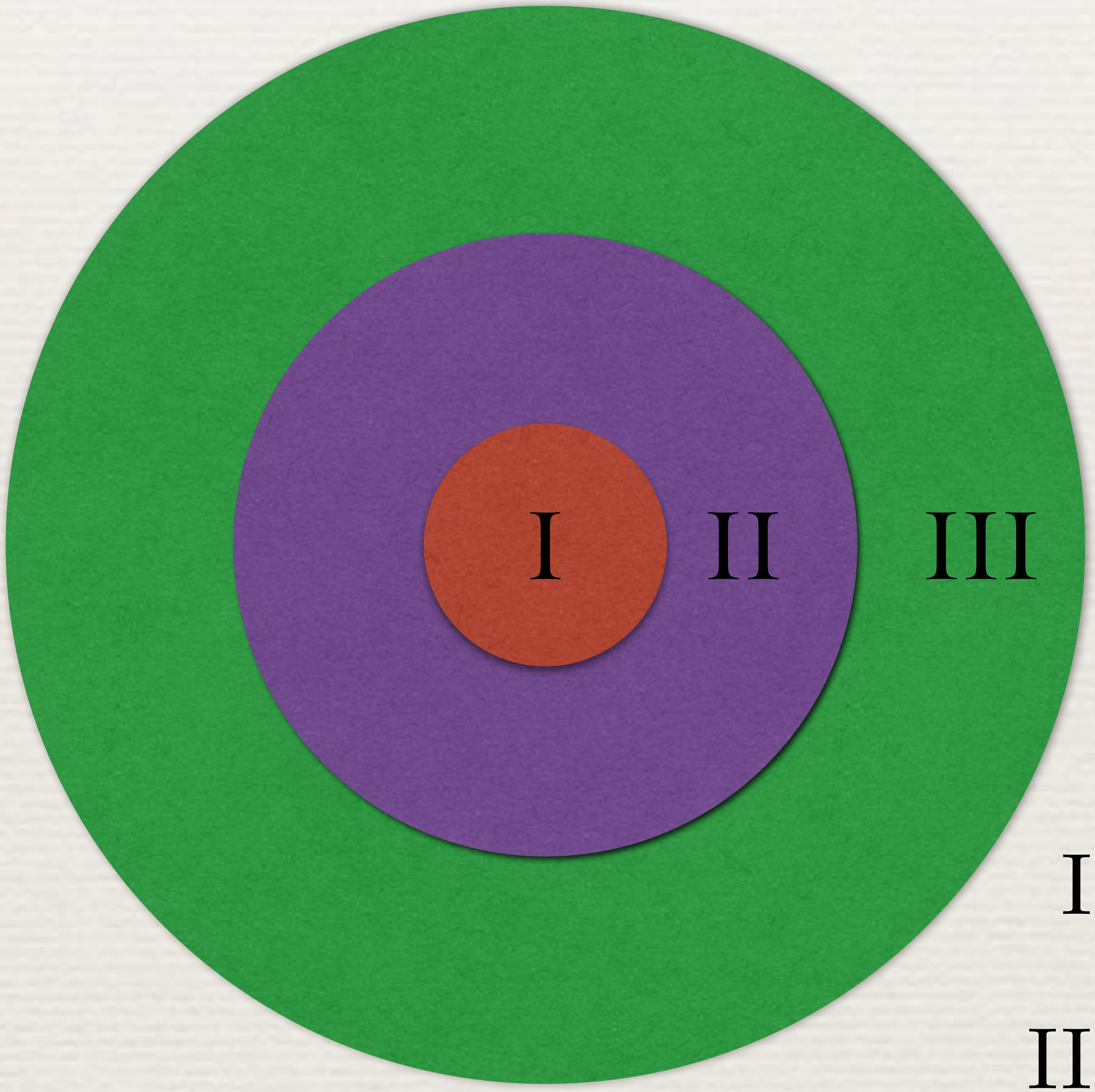
$$\lim_{\rho\to\infty}\rho\left(\frac{\partial}{\partial\rho}+ik\right)\mathbf{E}(\rho,\theta)=0$$

thus

$$\mathbf{E}=\tilde{\mathbf{E}}e^{-ik\rho},\quad \tilde{\mathbf{E}}=0(1/\rho)$$

- assumption $\tilde{\mathbf{E}}$ is a smooth function in $s=1/\rho$ in the vicinity of infinity

Multi-domain approach



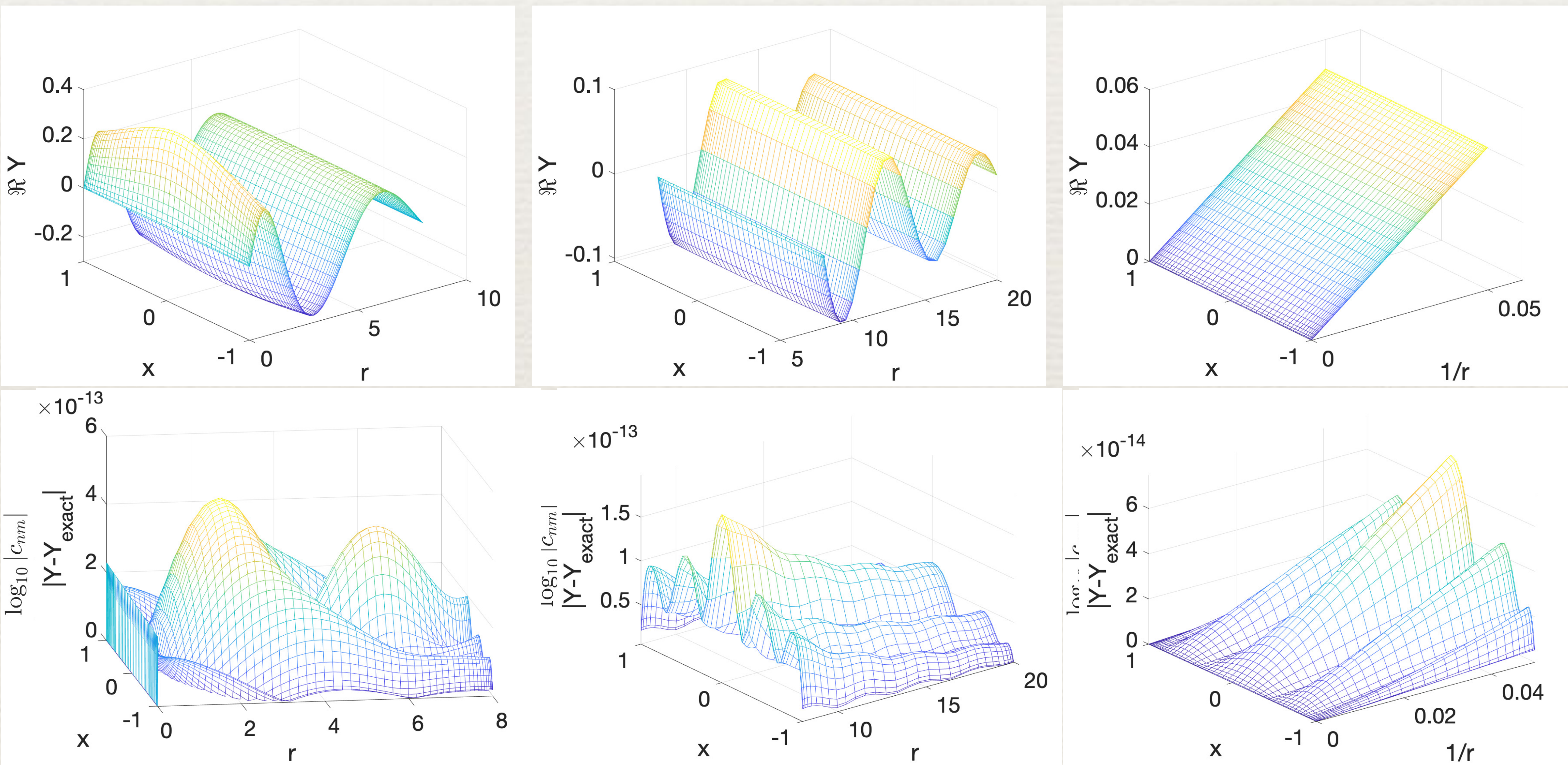
I $\rho = r_I(1+l)/2$

II $\rho = r_I(1-l)/2 + r_{II}(1+l)/2$

$l \in [-1, 1]$

III $\rho = 2r_{II}/(1+l)$

Example



Benjamin-Ono equations

C. Klein, J. Riton, N. Stoilov, Multi-domain spectral approach for the Hilbert transform on the real line, SN Partial Differential Equations and Applications (2:36) (2021) https://doi.org/10.1007/s42985-021-00094-8

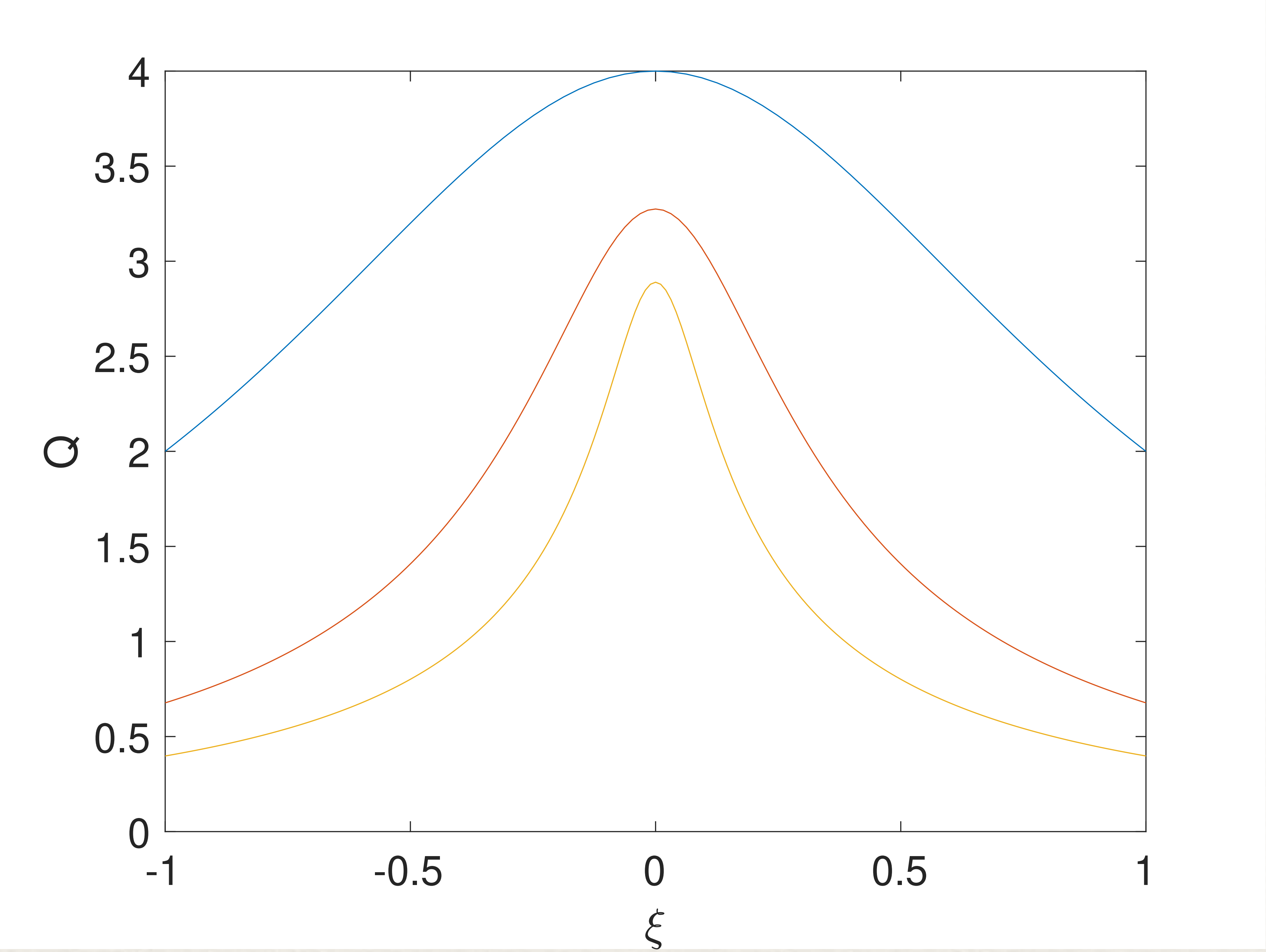
- $$u_t + u^{m-1}u_x - \mathcal{H}u_{xx} = 0,$$
- Hilbert transform

$$\mathcal{H}[f](x) := \frac{1}{\pi}\mathcal{P}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy$$

- solitary wave solutions

$$-cQ_c(\xi) - HQ_c'(\xi) + \frac{1}{m}Q_c^m(\xi) = 0$$

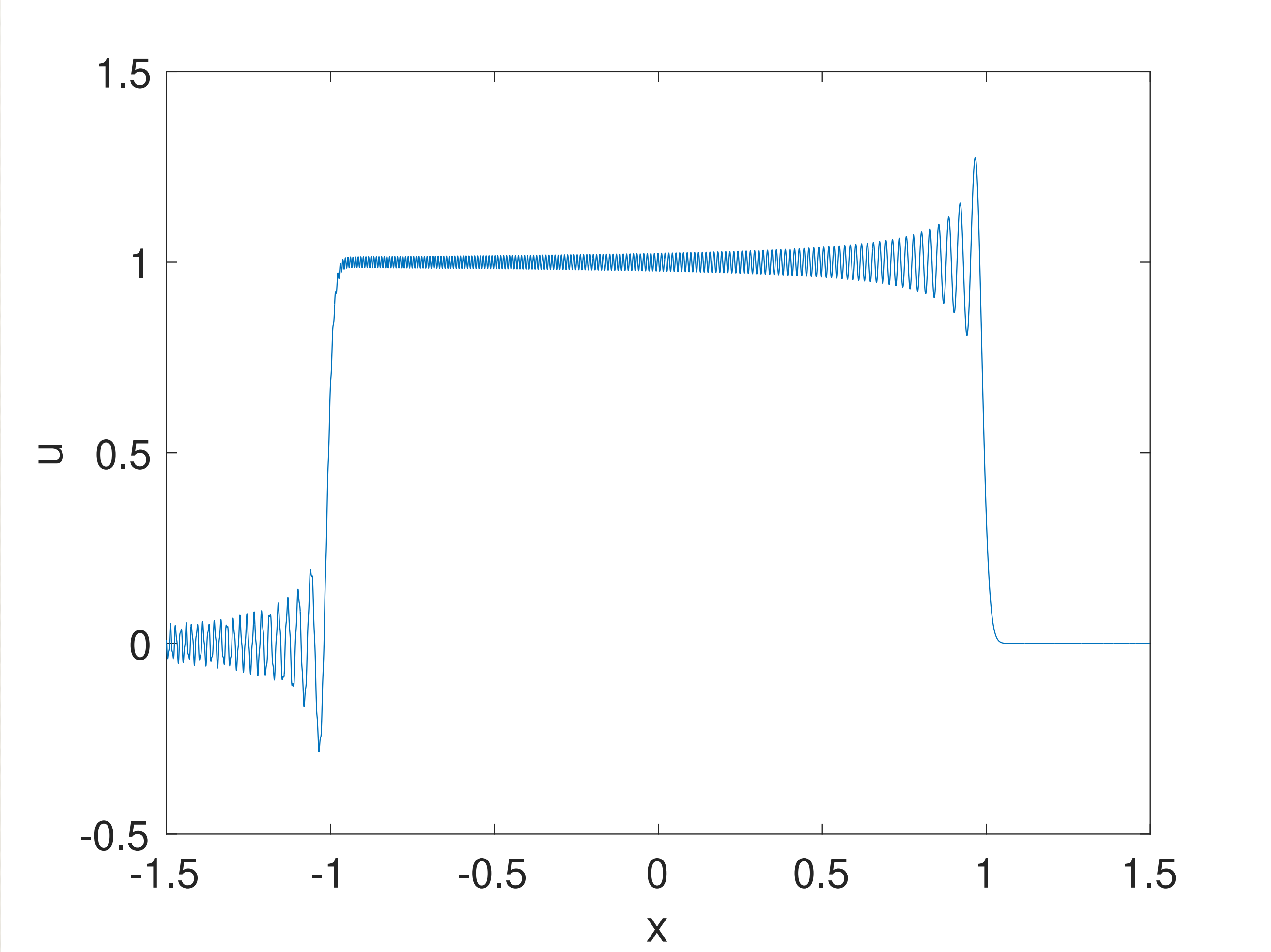
Newton iteration



Fourier transforms

- diagonal differentiation matrices, efficient for time integration
- not well approximated by DFT for slowly decreasing and discontinuous functions
- integration in the complex plane on contours motivated by steepest decent

Step initial data for Airy equation



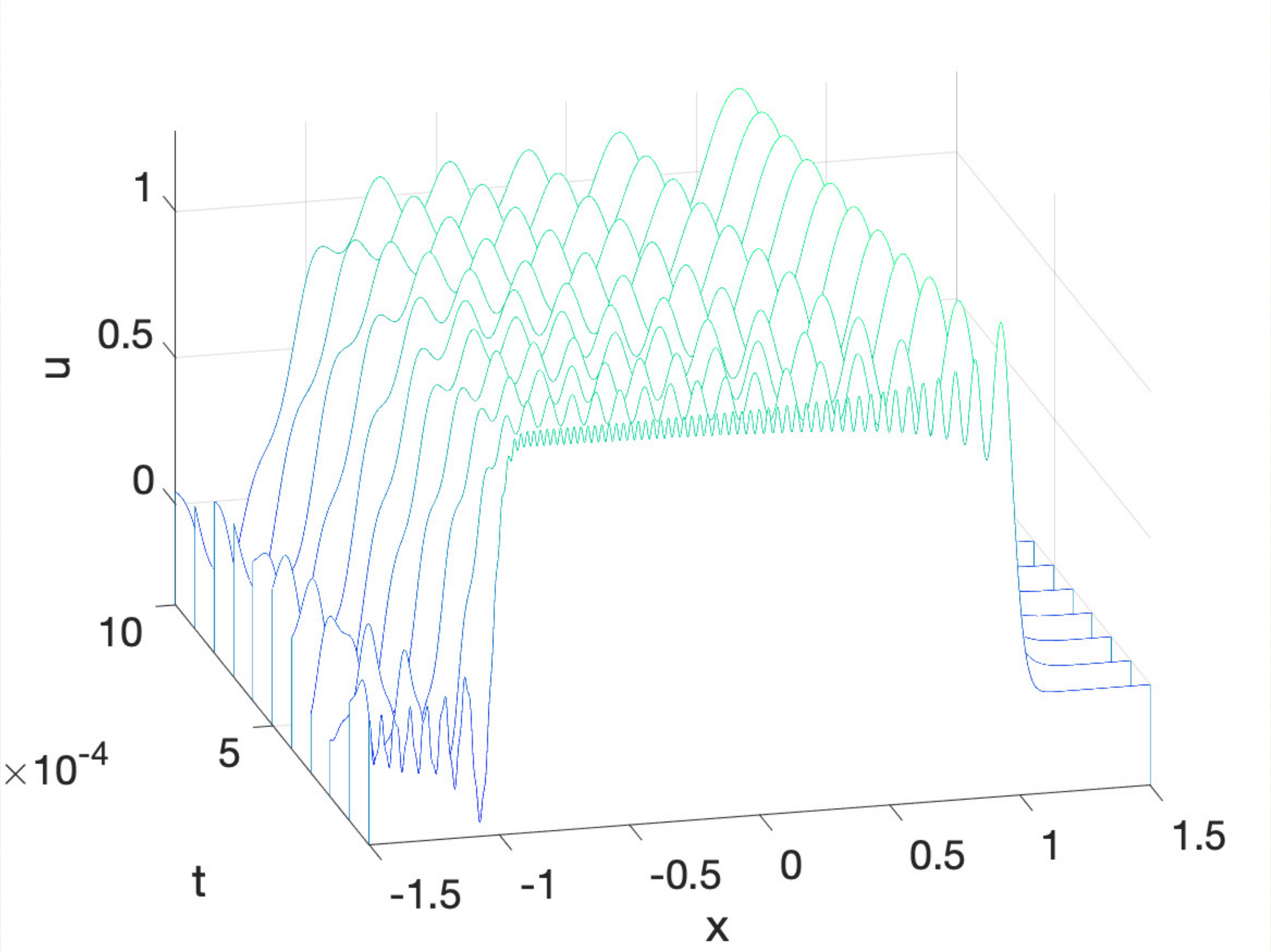


FIGURE 5. Solution for the Airy equation (2) and step initial data (7): on the left for $t = 10^{-6}$, on the right for several values of $t \in [10^{-5}, 10^{-3}]$.

Integration path

$$F(\eta) = \int_{\mathbb{R}} \frac{dk}{k} \exp(ik^3 + ik\eta)$$

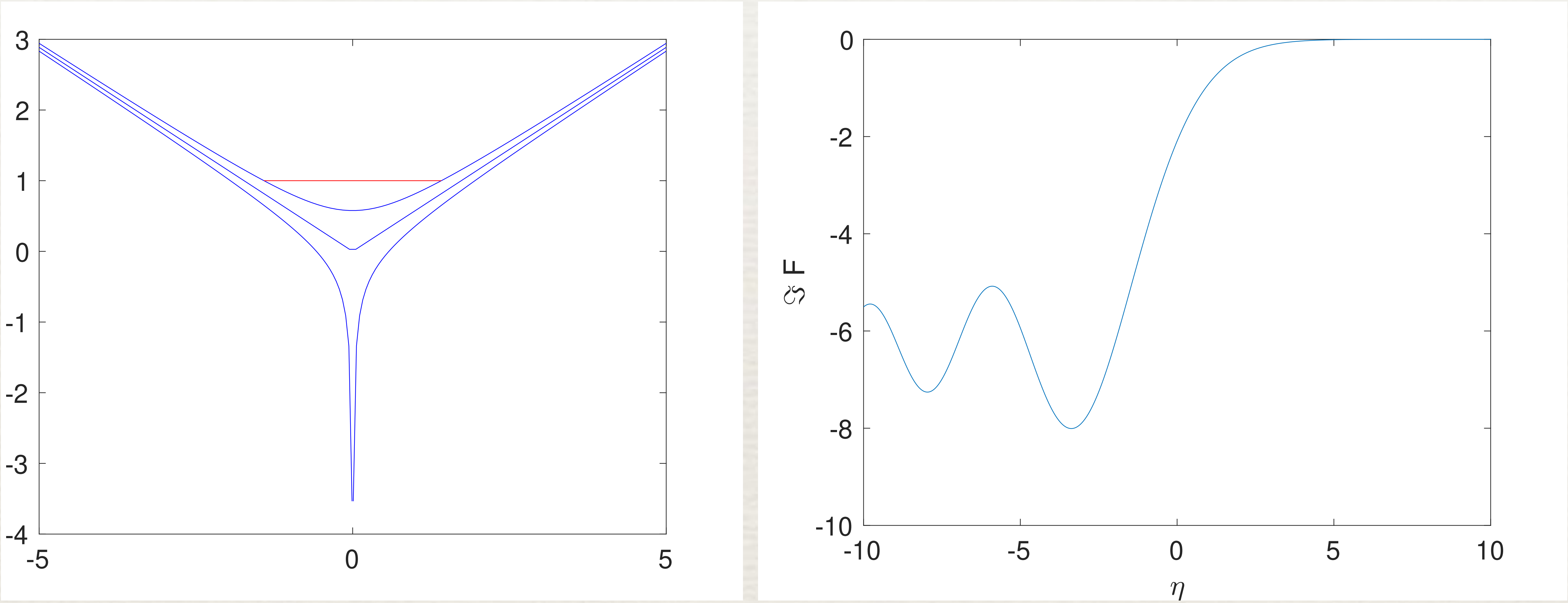


FIGURE 6. On the left the integration contours in the complex k-plane for $\eta = 1$ (top), $\eta = 0$ (middle) and an open contour for $\eta = -1$ (in red the contour bridging between the blue arcs for $\eta = 1$ to address the pole at the origin), on the right the function $F(\eta)$.

Outlook

- adapted time integrators
- matching conditions for higher order PDEs
- fractional derivatives
- Fourier transforms

References

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